The Optimal Allocation of Securities for Portfolios

Junhao Zhang

ABSTRACT

Portfolio optimization assists in the selection of the optimum portfolio to meet certain goals. The most often used portfolio optimization model is the Markowitz Model (MM). The approach highlights the need to select assets complementing one another to reduce risk for investors. It compares the risks and returns of multiple equities to discover which asset offers the best returns while posing the fewest hazards. To simplify the Markowitz Model, the Index Model (IM) employs a single element, the market index, which impacts all investment returns. Using the MM and IM, this study analyzes permitted portfolio areas for ten stocks and one broad equity index. The ten businesses were chosen from various industry areas. SPX, NVDA (Technology), CSCO (Technology), INTC (Technology), The Goldman Sachs Group (Financial Services), US Bancorp (Financial Services), TD CN (Financial Services), Allstate (Financial Services), Procter & Gamble Company (Personal Care Products), Johnson & Johnson (Pharmacy), Colgate-Palmolive (Personal Care Products).

The firms from several industrial sectors are chosen to guarantee the risk-diversified final portfolio. As a consequence, the efficient portfolio employs the weights that yield the highest returns for a given risk level or the lowest risk for a given projected return level when determined using the MM or IM. We concluded from the data that the IM optimization model well approximates the MM optimization model by minimizing the number of estimations necessary for model estimation.

Keywords: Markowitz model, Index model, Normal distribution, optimal portfolio, minimal risk portfolio

methodology

The Full Markowitz Model ("MM")

Assumptions:

1. Investors evaluate each investment decision Based on the probability distribution of securities returns throughout a specific position.
2. Based on the variance or standard deviation of the expected return of the investment, the investor calculates the risk of a portfolio.
3. The security's risk and return drive the investor's choice.
4. The investor optimizes expected return for a given level of risk or minimizes portfolio risk for a given level of expected return.

The MM model portfolio P's anticipated return is:

$$R_p = \sum_{i=1}^{n} w_i r_i$$

Mean: The expected return on the portfolio is the weighted average of the projected returns on the securities.

The Standard deviation of Portfolio P is:

$$\sigma_p = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j cov(R_i, R_j)}$$

r_i: the expected return on Asset i,

w_i: represents the proportion of asset i in the portfolio

n: the number of total assets,

COV(Ri, Rj): represents the covariance between the return on asset i and the return on asset j.

Single Index Model("IM")

Assumptions:

Two assumptions in Willieam Sharpe's single Index model.

1. Securities are subject to two types of risk: idiosyncratic and systematic. Unsystematic risk is unaffected by variables like indexes.
2. The idiosyncratic risk of one asset does not affect the idiosyncratic risk of another security, and the only way that the returns of the two securities are associated is through the combined response of the variables.

The previous two assumptions indicate that cov(R_m,_i)=0; cov(_i,_j)=0;; which greatly simplifies the computation.

The expected return of the IM model portfolio p is

$$R_p = \sum_{i=1}^{n} w_i r_i$$

The Standard deviation of Portfolio P is:

$$\sigma_p = \sqrt{\left(\sum_{i=1}^{n} w_i \beta_i\right)^2 \sigma_M^2 + \sum_{i=1}^{n} w_i^2 \sigma_\epsilon^2}$$

r_i: the expected rate of return of Asset i

w_i: denotes the proportion of asset i in the portfolio
n: the number of total assets,

β_i: the risk factor of asset i

σ_M: systematic risk

σ_ϵ: unsystematic risk

Then, we use M to express the SPX index, so the Excess rate of Return is

$$R_M = r_M - r_f$$

Then, we regress the security excess return Ri onto the RM using historical data for Ri(t) and RM (t), with t being the date of the observed sample, to get the regression equation:

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t)$$

Comparison Object

We define $r_p$as as the rate of return of risk portfolio P

$$riskpremuim = E_{(rp)} - r_f$$

Then we suppose the risky asset percentage is y and the risk-free asset proportion is 1-y. Portfolio C's rate of return is:

$$r_c = y_m + (1-y) r_f$$

The **expected return rate** is:

$$E_{(rc)} = yE_{(rp)} + (1-y) r_f = r_f + y\left[E_{(rp)} - r_f\right]$$

Then we combine the risky and risk-free assets to form a full portfolio C, and its standard deviation is:

$$\sigma_C = y\sigma_p$$

$$y = \frac{\sigma_c}{\sigma_p}$$

In this section, we introduce the idea of the CAL (Capital Allocation Line), which is a line that describes a mix of risky and risk-free portfolios for various values of y. The Sharpe Ratio is the name given to the CAL's slope. The expected return is expressed as a ratio of the risk, as follows:

$$S = \frac{E_{(R_D)} - r_P}{\sigma_C}$$

Minimum-Variance Frontier: The frontier is the curve drawn by the portfolio point with the lowest variance for a certain predicted return for the portfolio. All individual assets are to the right of this line beyond.

$$\begin{cases} \sigma(\vec{w}) \to min(\vec{w}) \\ subjectto: r(\vec{w}) = const \end{cases}$$

Minimal Return Frontier:

$$\begin{cases} r(\vec{w}) \to min(\vec{w}) \\ subjectto: \sigma(\vec{w}) = const \end{cases}$$

Efficient Frontier: Because they offer the best risk and return, all of the points on the minimal variance horizon above the least variance portfolio may be regarded as the best portfolio.

$$\begin{cases} r(\vec{w}) \to max(\vec{w}) \\ subjectto: \sigma(\vec{w}) = const \end{cases}$$

Global Minimal Risk Portfolio: The global Minimum-Variance frontier:

$$\left\{\sigma(\vec{w}) \to min(\vec{w})\right.$$

Optimal Risky Portfolio: The tangential point of the efficient frontier and CAL has a Maximal Sharpe Ratio, which indicates it has the highest return and the lowest variance.:

$$\left\{\frac{r(\vec{w})}{\sigma(\vec{w})} \to max(\vec{w})\right.$$

Normal Distribution

The deviation of an empirical data distribution from a normal distribution is often assessed using the following:

$$Skewness = \frac{(x-\mu)^3}{\sigma^3}$$

$$Kurtosis = \frac{(x-\mu)^4}{\sigma^4} - 3$$

Comparison with normal distribution

Statistical Comparison

The Markowitz Method formulas we used are as follows.
The Markowitz Model (MM) expected portfolio return:

$$r_p = \vec{w} * \vec{\mu}^T$$

Markowitz Model (MM) investment portfolio expected standard deviation:

$$\sigma_p = \sqrt{\vec{v}} P \vec{v}^T$$

The Index Model (IM) formulas we used are as follows.
The expected portfolio return following the Index Model:

$$r_p = \vec{w} * \vec{\mu}^T$$

The Index Model investment portfolio expected standard deviation:

$$\sigma_p = \sqrt{(\sigma_M \beta_p)^2 + \sum_{◆=1}^{n} w_◆^2 \sigma^2(\epsilon_◆)},\ \beta_p = \vec{w} * \vec{\beta}^T$$

The following are some of the Markowitz Model's drawbacks:

1. Because of the excessively high number of estimates needed for big portfolios, MM cannot be used for such

portfolios.

2. If one naively thinks that we must obtain the predicted estimates from the estimation of historical data, then this adds reliance on the sample size and sampling frequency. All of this is not included in the first MM. This is why IM has advanced and gained such a user following.

Markowitz Model's Description

We define $\vec{\mu} = \{\mu 1, \mu 2, \mu 3, ...\mu n\}^T$ is the set of instruments' average returns; $\vec{w} = \{w1, w2, w2, ... wn\}^T$ is the unknown set of instruments' weights; $\vec{\sigma} = \{\sigma 1, \sigma 2, \sigma 3, ...\sigma n\}^T$ is the set of instruments' standard deviations; $\vec{\beta} = \{\beta 1, \beta 2, \beta 3, ...\beta n\}^T$ is the set of instruments' betas; $\{\sigma(\epsilon 1), \sigma(\epsilon 2), \sigma(\epsilon 3), ...\sigma(\epsilon n)\}^T$ is the set of the residuals' standard deviations; $\vec{v} = \{\vec{v1}, \vec{v2}, \vec{v3}, ...\vec{vn}\}^T$ is an auxiliary vector; and $\begin{cases} \rho 11 & \rho 12 & .... & \rho 1n \\ \rho 21 & \rho 22 & ...... & \rho 2n \\ \rho n1 & \rho n2 & .... & \rho nn \end{cases}$ p= is the matrix of instruments' cross-correlation coefficients.

The Markowitz Model may be used for real-world asset allocation problems. To diversify investment risk and maximize investment utility, the investment percentage of securities may be changed by establishing the optimal asset portfolio. The risk reduction provided by diversification through using low-correlation assets in a portfolio is a significant feature suggested by this model.

Capital Market Lines

Capital Market Line (CML) is an upward-sloping line that symbolizes the risk-return trade-off in the capital market and implies that an investor will take on more risk if the portfolio's return is likewise higher.

Characteristics of CML:

- It is the ideal mixture of hazardous investments at the tangent point P to the efficient frontier.
- Only efficient portfolios, the market portfolio P, and portfolios made up entirely of risk-free assets are allowed on the CML.
- CML has a constantly increasing slope since the cost of risk must be positive.
- Rational investors will only invest if they are promised compensation for the risk.

Introduction to Formulas

• Variance:

$$\sigma^2 = \sum (R_j - E(R_j))^2 * P_j$$

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum (R_j - E(R_j))^2 * P_j}$$

• Covariance:

$$Cov = \frac{1}{m-1} \sum \left[R^A{}_j - E(r)^A \right] [R^B{}_j - E(r)^B]$$

• Portfolio Return:

$$R_p = \sum_{t=1}^{N} X_n R_n$$

R_P: Return on the portfolio

X_n: proportion of portfolio return investment in security n

R_n: Expected return on security n

• Capital Market Line Calculation Formula:

$$R_p = \frac{I_{RF} + (R_M - I_{RF})\sigma_M}{\sigma_p}$$

R_p: the expected return of the portfolio

R_m: return on the market portfolio

I_{RF}: risk-free rate of interest

σ_m: standard deviation of the market portfolio

σ_p: standard deviation of portfolio

• Index Model Calculation Formulas:

$$R_{it} - R_f = \alpha_i + \beta_i (R_{mt} - R_f) + \varepsilon_{it}$$

$$\varepsilon_{it} \backsim N(0, \sigma_i^2)$$

The above formula defines the index model. The variables i and i represent the stock's alpha or anomalous return and beta or responsiveness to the market, respectively, are perhaps two of the most important quantities. The residual return, considered independent and normally distributed with a mean of zero and a standard deviation of i, is also known as Rit- Rf, the excess return of the stock, and Rmt-Rf, the excess return of the market.

We chose ten stocks and one market index from various countries, industries, and sectors to create a portfolio with a wide range of holdings.

DATA-PROCESSING

Data Description

To put the model hypothesis to the test, we select ten companies from three distinct equities sectors—technology, financial services, and industries and we use the S&P 500 as both a market index (totaling 11 risky assets) and a stand-in for a risk-free rate (the previous month's federal funds rate). Using Bloomberg Professional, we obtained daily data for these equities from May 11, 2001, through May 12, 2021. We then processed the data further to obtain the corresponding monthly data, using only five working days of daily data per week. More particular information about the ten stocks is provided below.

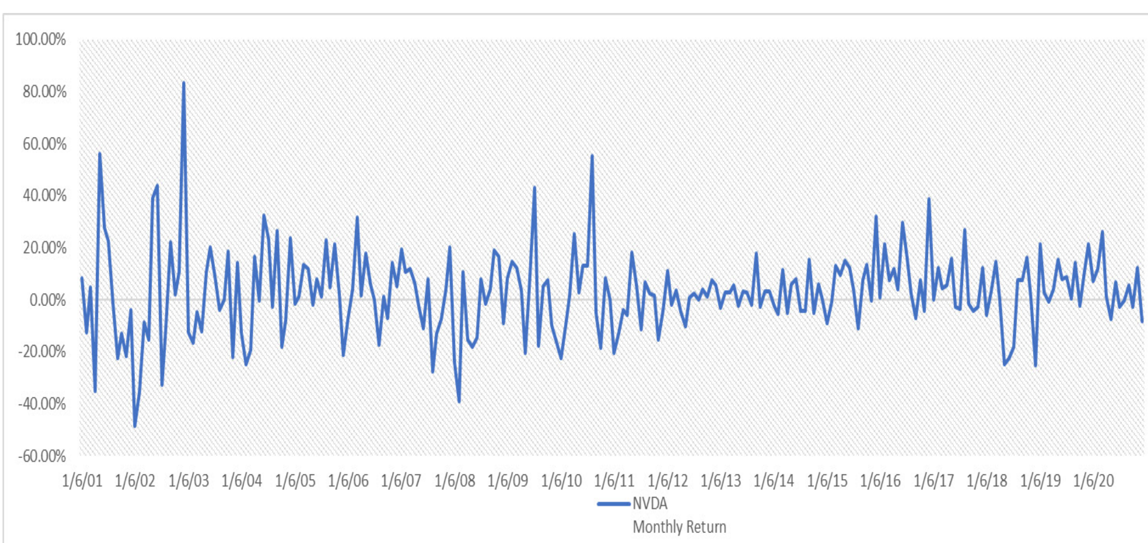
Selection of Stocks

Nvidia Corporation

NVIDIA is a fabless semiconductor company that provides system-on-chip (SoC) units for mobile computing and automotive sectors and graphics processing units (GPUs) for gaming and professional applications. It focuses on GPU architecture to provide platforms for 3D internet applications, robotics, self-driving cars, artificial intelligence, data research, and the metaverse. GPUs, laptops, G-SYNC displays, workstations, GeForce graphics cards, embedded systems, and data center solutions are all available from the firm. NVIDIA also develops infrastructure suites, cloud services, gaming software, applications, and tools.

Picture-1

Chart-1



The variations in Nvidia return were flatter, as illustrated in the monthly-return figure. Investors would rank Nvidia as an investment based on the lower expected P/E and the smoother volatility.

Reference:

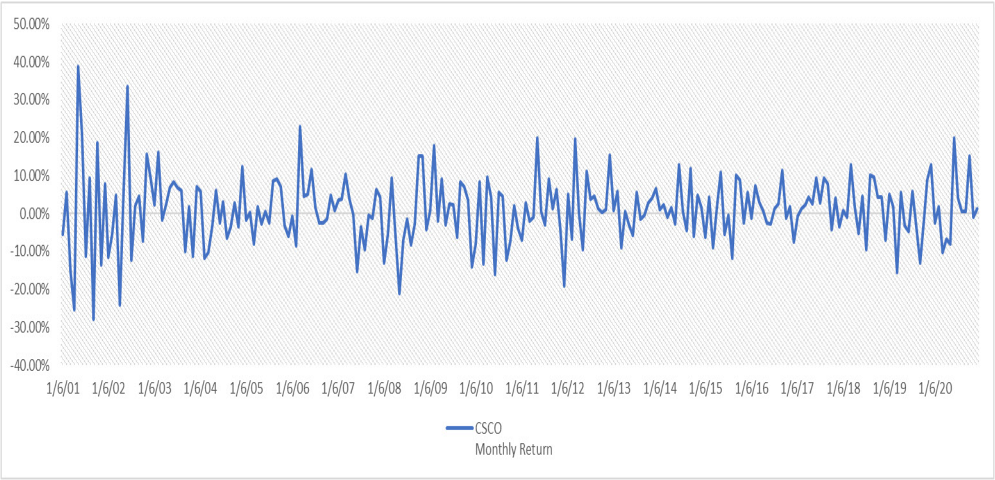
NVIDIA Company Profile - Office Locations, Competitors, Revenue, Financials, Employees, Key People, Subsidiaries | Craft. co

Cisco

Cisco offers an industry-leading assortment of technology breakthroughs that help communities and companies connect safely through networking, security, collaboration, cloud management, and more. However, We understand that hardware is challenging because Cisco is attempting to shift away from it in favor of faster-growing, more consistent, and higher-margin software and subscription income. Currently, Cisco is increasing its internal development. It has introduced its Security Cloud, which integrates cybersecurity with its historical hardware offering. AppDynamics Cloud assists businesses in managing cloud infrastructure.

Picture-2

Chart-2



Despite intermittent losses for investors throughout the past 15 tears, we could see that Cisco had significant, steady swings. Specifically, IT demand increased during the Covid-19 period because it had to. The increase of remote work and the rush into e-commerce by many brick-and-mortar businesses necessitated massive infrastructure investment, enhancing industry profitability. Cisco could not capitalize on such demand and strengthen investors' confidence in the forecast price due to the market price continuing to grow but lower predicted P/E.

Reference:

About Cisco - Cisco

https://www.investing.com/analysis/cisco-stock-is-cheap-for-very-good-reason-200626173?utm_source=google&utm_medium=cpc&utm_campaign=18408068078&utm_content=646224881948&utm_term=dsa-1547773562090_&GL_Ad_ID=646224881948&GL_Campaign_ID=18408068078&ISP=1&ppu=1&gclid=EAIaIQobChMImum-wOPO_wIVsNlMAh2wLg-UEAAYBCAAEgLeCvD_BwE

Intel Corporation

Since its inception in 1968, Intel has produced groundbreaking computer technology and has been a market leader in technology that alters the world and improves people's lives. Artificial intelligence (AI), the growth of 5G networks, and the rise of the intelligent edge are three technical tipping points that will impact technology's future. These shifts are being driven by hardware and software, with Intel at the epicenter of it all.

Picture-3

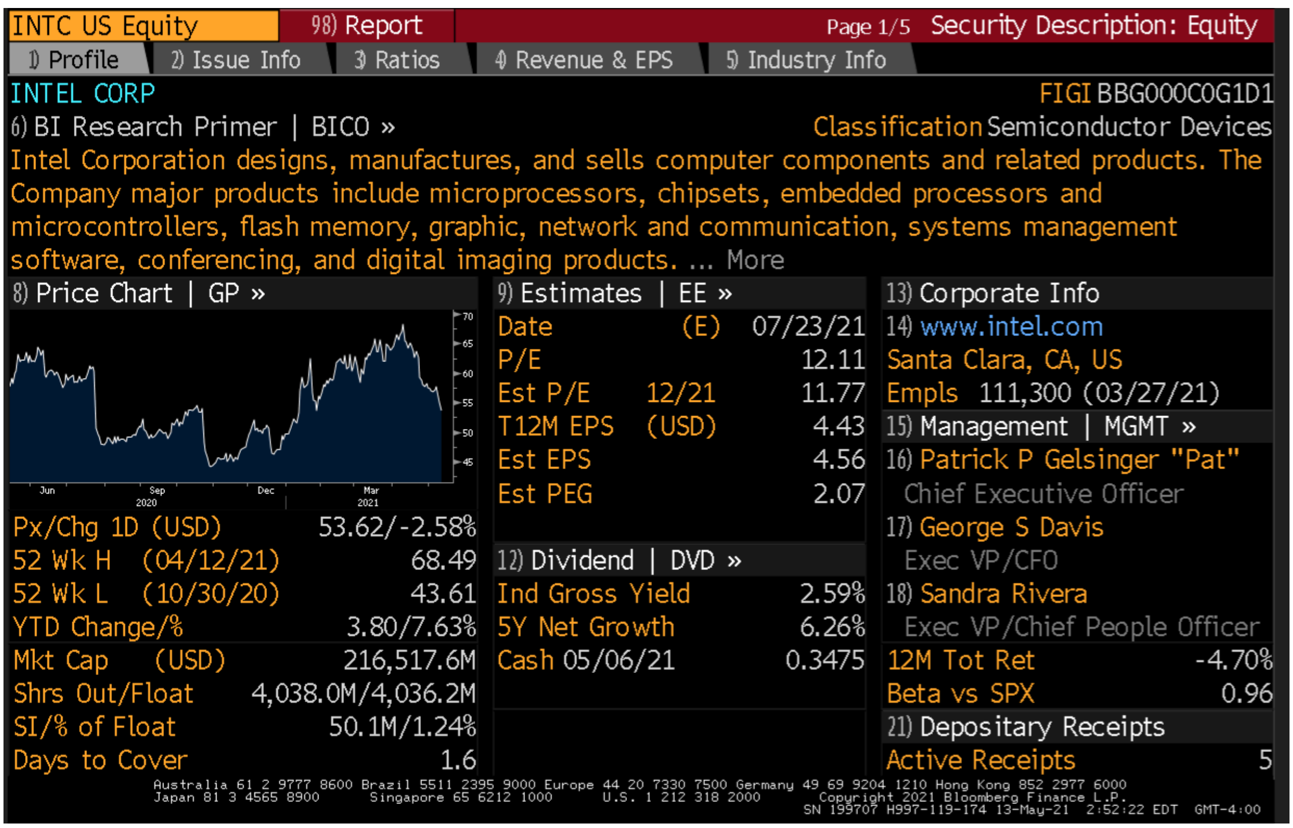
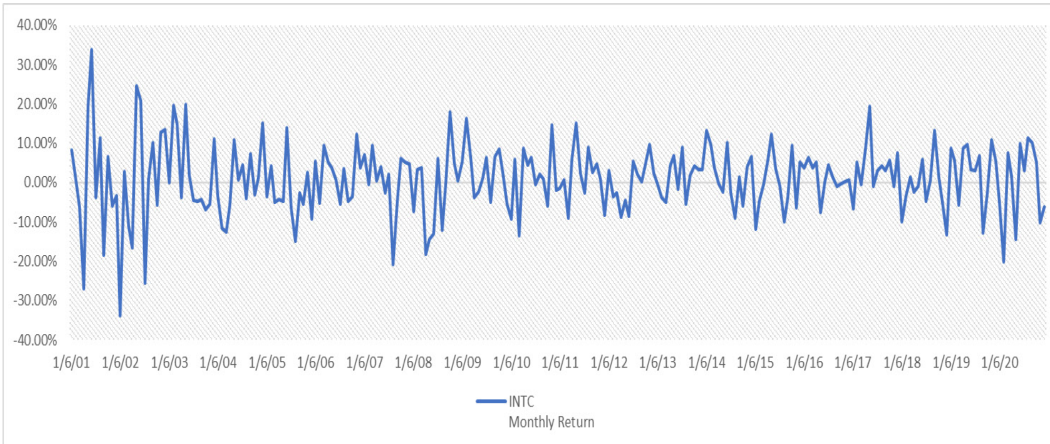


Chart-3



Even if the network and technology advanced quickly from 2018 to 2019, the Intel Corporation showed significant variations. The Intel scenario is markedly different from CSCO's, with a negative price shift since 2021 and a lower 5-year net growth estimation. Moreover, the median price objective for Intel Corp from the 33 famous analysts from Wall Street providing 12-month price estimates is 31.00, with a high estimate of 45.00 and a low estimate of 17.00. The median forecast is -14.88% lower than the previous price of 36.42.

Reference:

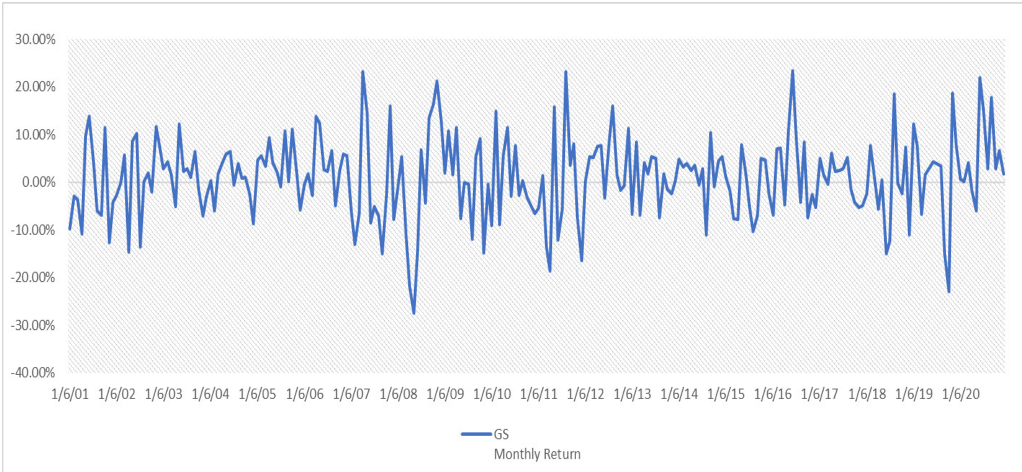
About Intel:: Intel Corporation (INTC)

GOLDMAN SACHS GROUP, INC.

The Goldman Sachs Group, Inc. is a worldwide financial organization that provides a wide variety of financial services to a diverse client base, including businesses, financial institutions, governments, and people. The business, founded in 1869, is headquartered in New York and has offices in all major financial cities worldwide.

Picture-4

Chart-4



According to a study, the Goldman Sachs Group, which possesses loans and securities with significant leverage, has seen continual ups and downs over the previous three decades. According to the chart, GS has suffered a significant loss since the Covid-19 spread, when the global economy was in decline. However, thanks to diverse investments and unique insights, GS has risen from the bottom to the top in months while maintaining high EPS and a stable P/E ratio.

US BANCORP

US Bancorp (USB) is a diversified financial services firm with subsidiaries that provide retail and commercial banking, private banking, and wealth management solutions. Savings and checking accounts, certificate of deposits, consumer and business loans, personal and business lines of credit, mortgages, insurance, savings, and investment products, brokerage and fund services, credit and debit cards, asset and wealth management, and financial planning solutions are all part of its product and service portfolio. The organization also offers leasing, international banking, payment services, private banking, cash management, and online and mobile banking. It primarily serves the Midwest and West areas of the United States. USB is based in Minneapolis, Minnesota, in the United States.

Picture-5

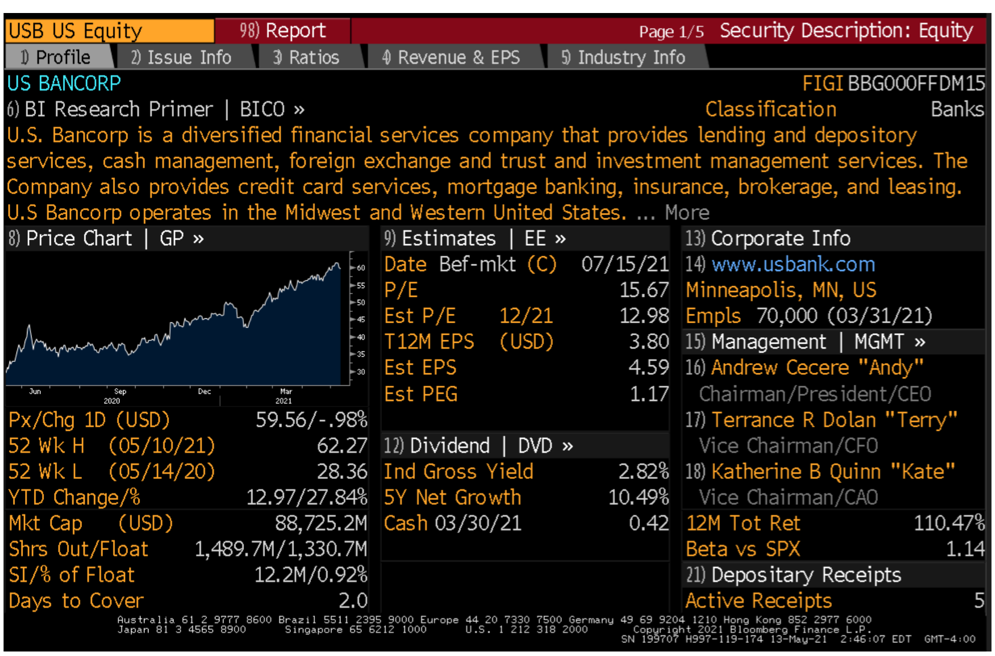
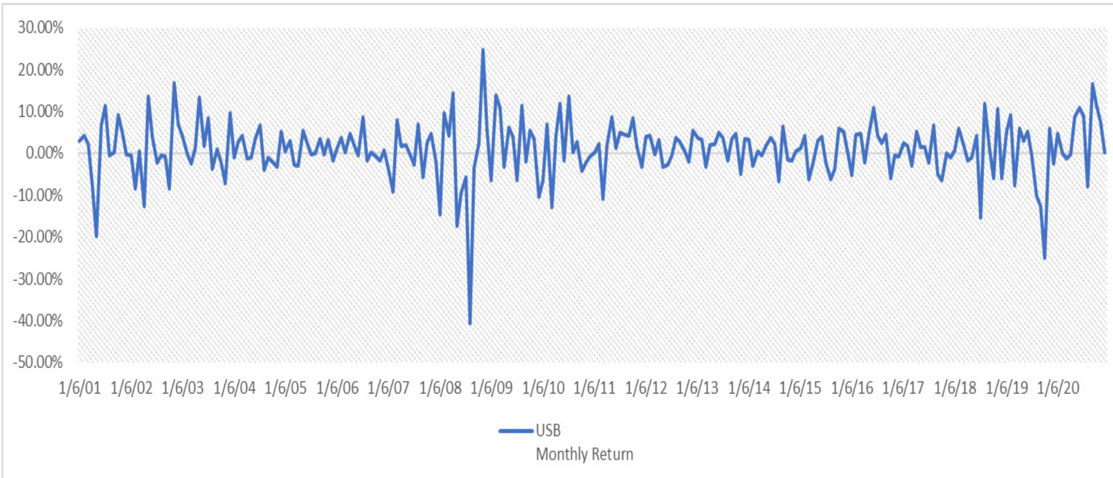


Chart-5



Although having stable 30-year returns, the US bank occasionally experienced particular losses with unexpected and immeasurable damage to investors. Similarly to GS, US Bancorp has climbed gloriously on the price chart and has greatly increased monthly returns from 2019 to 2022.

Reference:

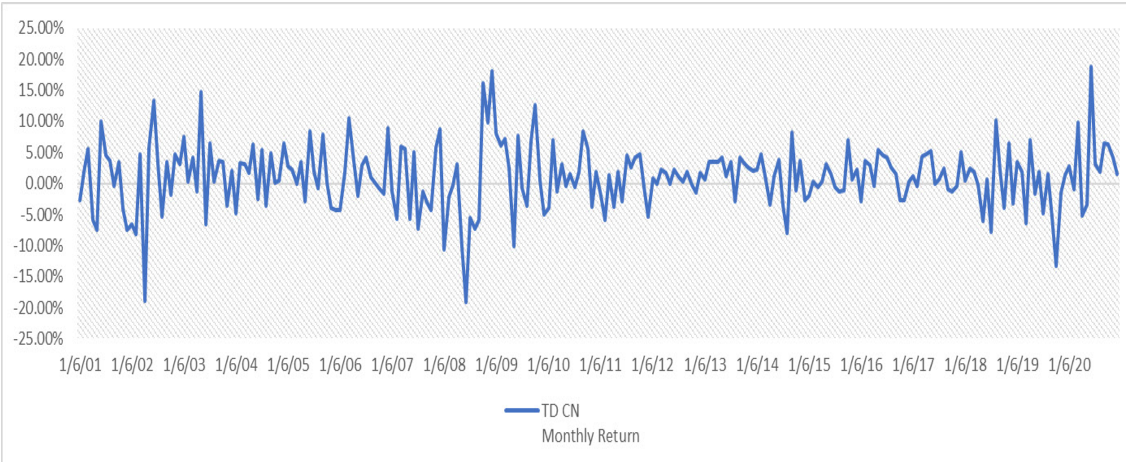
US Bancorp Company Profile - Overview - GlobalData

TD Bank

The Toronto-Dominion Bank, or TD, is Canada's second-biggest chartered bank. The Toronto-Dominion Bank was formed by amalgamating three financial institutions: The Bank of Toronto, The Dominion Bank, and Canada Trust. The first merger occurred in 1955 when The Dominion Bank combined with The Bank of Toronto. In 2000, this consortium purchased Canada Trust, becoming TD Canada Trust. Toronto-Dominion Bank is a public business that trades under the ticker TD on the Toronto Stock Exchange and the New York Stock Exchange. In 2022, TD had $49.03 billion in revenue, $17.43 billion in net income, and $1.92 trillion in assets. However, the global footprint of TD Bank is inadequate. The bank is more reliant on Canada and the United States. In addition, TD Bank trails behind the leading Canadian banks regarding branch count.

Picture-6

Chart-6



TD Bank periodically demonstrated excess positive returns but often had negative payouts over the previous 30 years. As a strong responder in the US and Canadian markets, TD has seen what GS and US Bancorp have done. On the other hand, TD bank cannot be completely safe in the face of unanticipated occurrences in the US market, such as rising federal rates.

Reference:

Toronto-Dominion Bank (TD) | The Canadian Encyclopedia

https://iide.co/case-studies/swot-analysis-of-td-bank/

ALLSTATE, CORPORATION.

Allstate was founded on April 17, 1931, as the Great Depression worsened and Americans battled financial instability. The initial coverage covered a 1930 Studebaker for $41.60 per year. The first claim was paid when a client strolled into Allstate's one-room headquarters carrying a door handle wrenched off by a would-be vehicle thief.

Picture-7

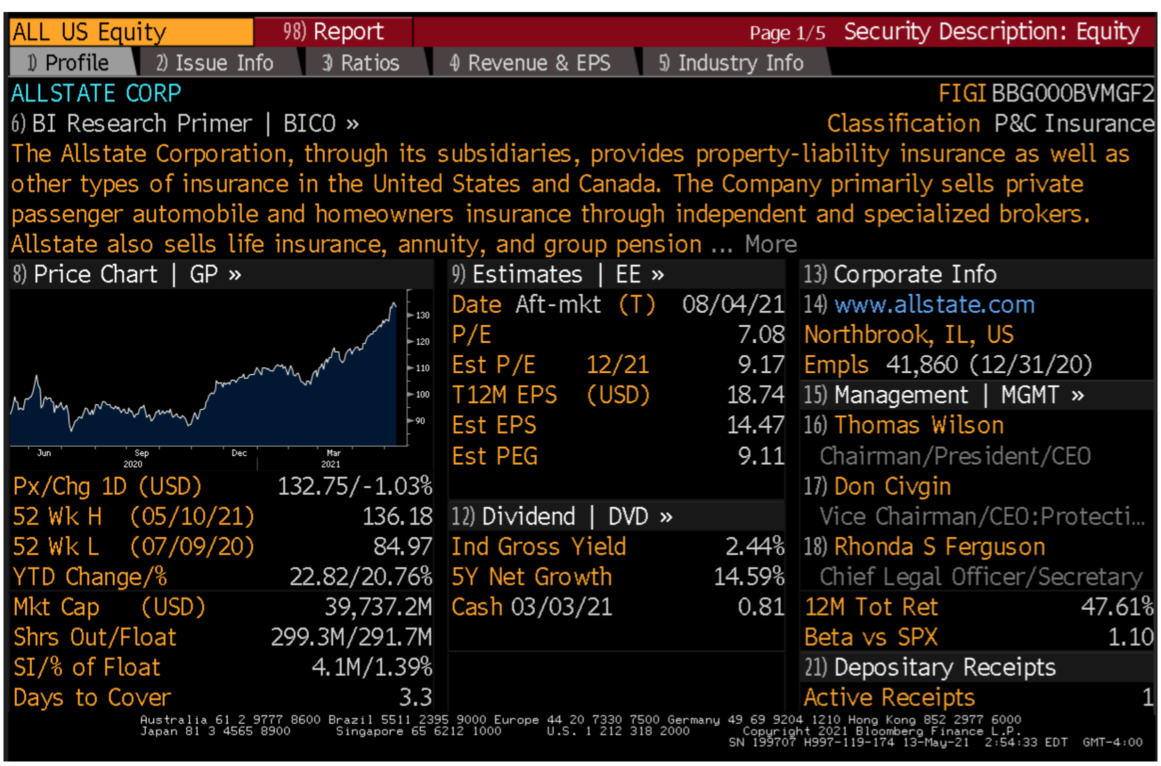
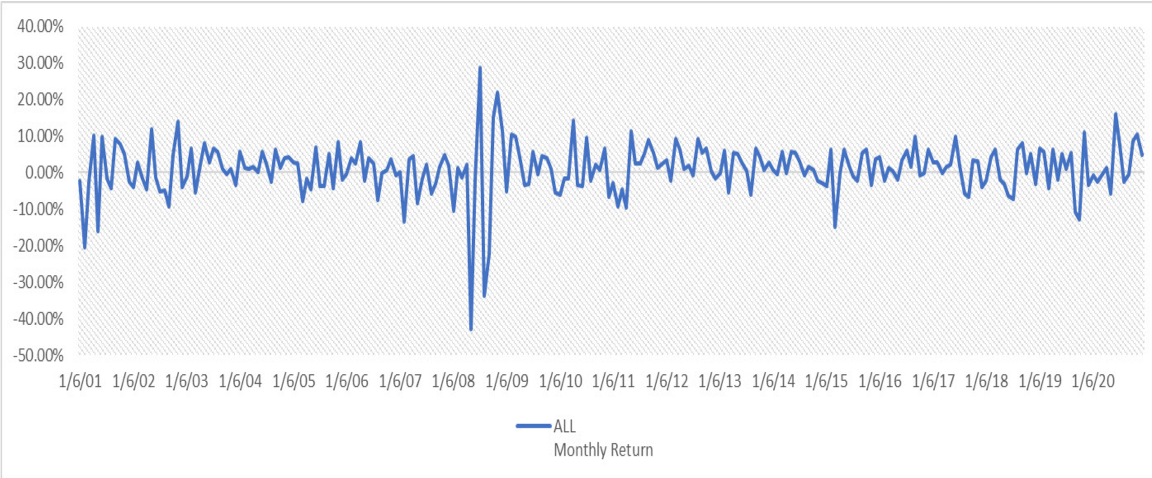


Chart-7



Allstate Corporation demonstrated a lower risk and smoother P/E projection than the other firms we chose. As an insurance company, Allstate was the only firm out of ten that did not experience substantial ups and downs during the covid-19 period.

Reference:

https://www.allstatecorporation.com/about/our-history.aspx#:~:text=Allstate%20launched%20on%20April%2017%2C%201931%2C%20as%20the,been%20pried%20off%20by%20a%20would-be%20car%20thief.

THE PROCTER & GAMBLE COMPANY

The firm, founded in 1837, furnished life supplies to the Union Army during the American Civil War. By the early twenty-first century, Procter & Gamble offered products in a variety of areas, including health and wellness, house and home, personal and beauty, baby and family. The company has long been a market leader and is continually developing new products.

Picture-8

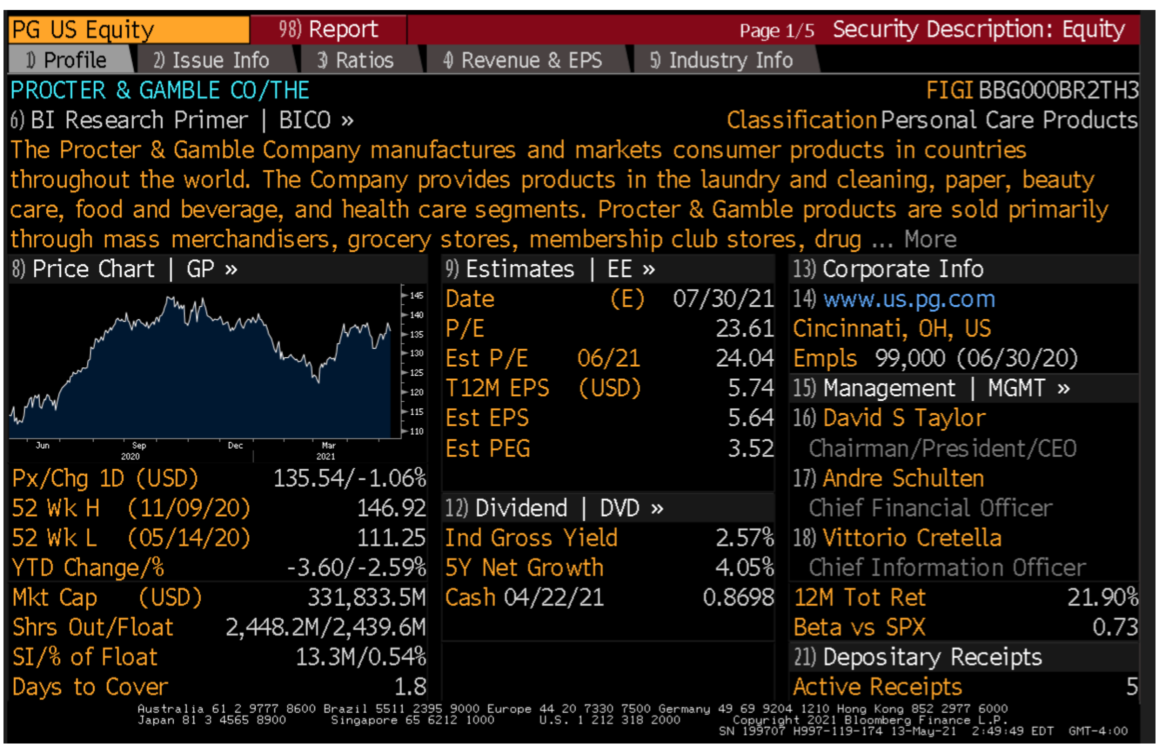
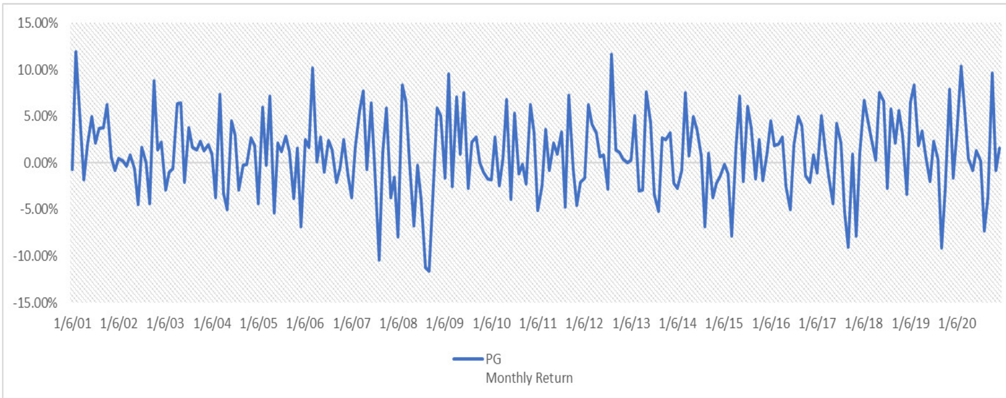


Chart-8



When combined with crucial ratios such as the P/E ratio and long-term swings, we discovered that losses were more frequent than gains and that the stock price was previously overestimated. During the Covid-19 period, the Procter & Gamble had tremendous global demand for its products, allowing it to increase its market share and sales in subsequent years.

Since the second part of fiscal 2022, the firm has proactively adopted price hikes to protect profits in the face of rising input prices and unfavorable currency moves. These pricing modifications have been critical in boosting the company's bottom line in recent quarters. P&G recently increased prices in February and March and aims to maintain this policy in the next quarters, depending on the macroeconomic backdrop in terms of input pricing and foreign exchange rates. This continued price increase implementation is likely to enhance the company's top line in the coming quarters.

Reference:

https://seekingalpha.com/article/4611681-procter-and-gamble-good-buy-current-valuation

Johnson & Johnson manufactures healthcare

Johnson & Johnson, with the goal of assisting in the creation of a better world, has specialized in research and development for a long time and has manufactured and sold a variety of health care goods. Consumer, Pharmaceutical, and Medical Devices and Diagnostics are the three business segments of the company.

Picture-9

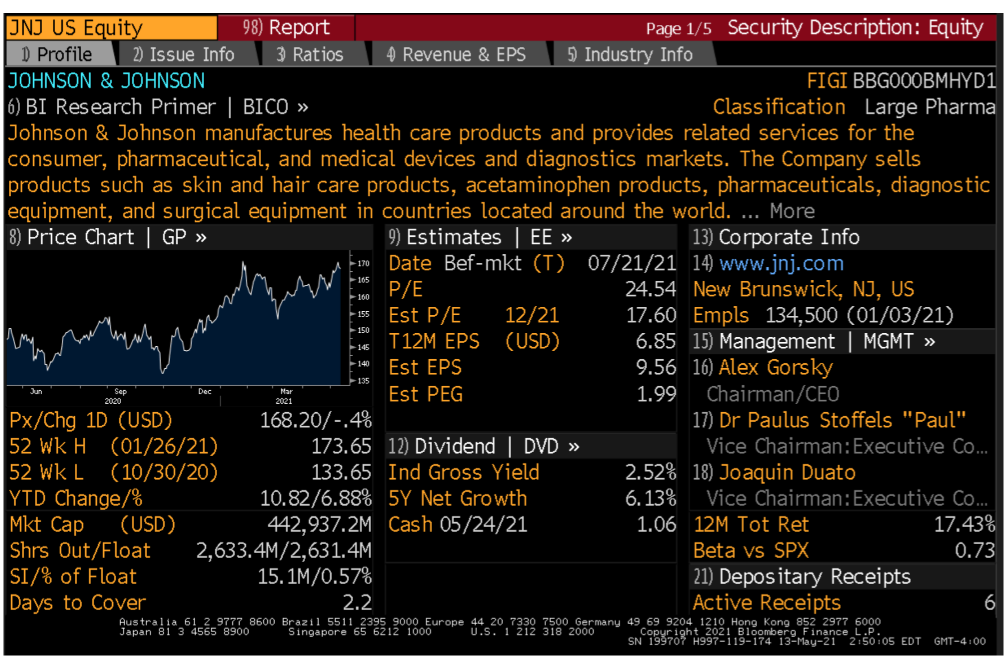
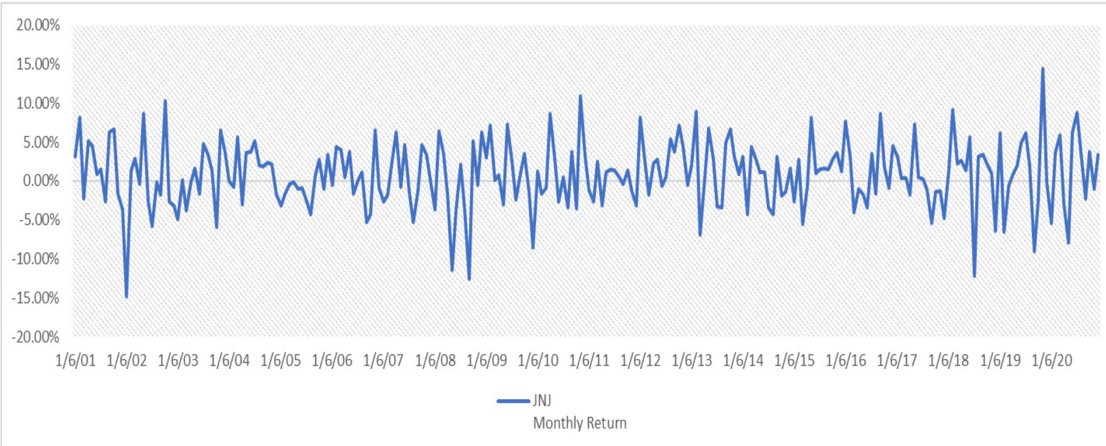


Chart-9



From 2016 to 2019, price movements were often negatively connected with variations, indicating that investors were not confidence in the stability of investment profit and that the stock price may fall, according to the estimated P/E ratio. The pharmacy business has also developed dramatically over the previous two decades, with worldwide pharma revenues predicted to reach 1.48 trillion US dollars in 2022, and JNJ, with a big market share, may be able to track the whole industry's growth. However, J&J has been at the heart of several scandals and government probes over the years, and the health-care behemoth has been the subject of numerous lawsuits.

Reference:

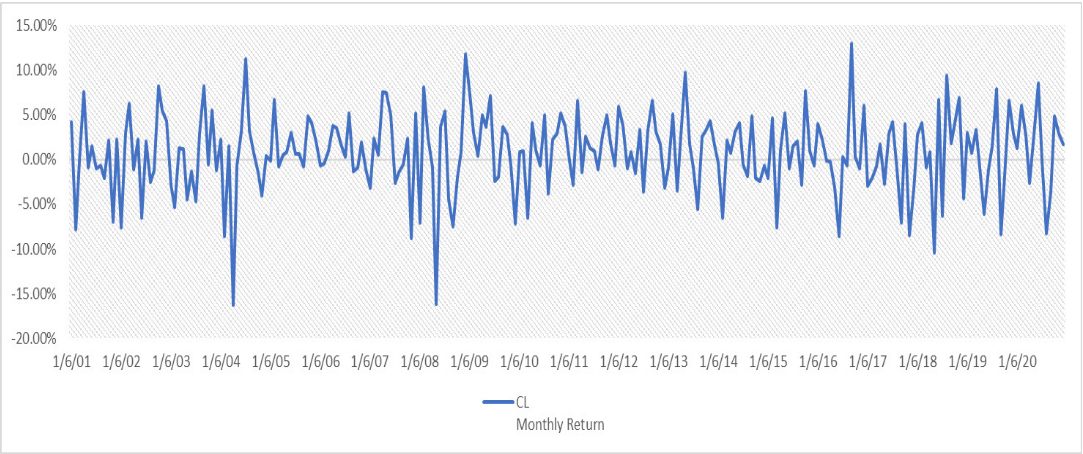
https://www.drugwatch.com/manufacturers/johnson-and-johnson/

COLGATEPALMOLIVE COMPANY

Colgate-Palmolive is one of the world's major makers of Fast-moving consumer goods (FMCG). The company's products are sold in over 200 countries and territories and are divided into four primary worldwide businesses: oral care, personal care, home care, and pet nutrition.

Picture-10

Chart-10



The monthly return of Colgate-Palmolive Company revealed an intriguing picture of substantial monthly rises in investment returns. Investors retained faith in the firm despite the innovation submission portal and market need for pharmaceuticals.
As an example. Deutsche Bank boosted its price objective on Colgate-Palmolive to $88 from $80 and maintained its Buy recommendation on the stock in 2023.

Calculation inputs and correlation test

Based on monthly data, we compute all of the needed estimates for each of the optimization problems MM and IM, and the results are displayed in Table-1.

Table-1 inputs results of the optimization problems

	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL
Annual Average Return	7.5%	32.8%	9.7%	8.9%	10.8%	9.9%	11.0%	10.1%	9.4%	8.5%	7.1%
Annual StDev	14.9%	55.8%	30.8%	30.5%	29.6%	23.7%	18.1%	24.9%	14.6%	14.8%	15.3%
beta	1.00	1.98	1.32	1.19	1.41	0.97	0.79	1.06	0.41	0.54	0.45
alpha	0.00	0.18	0.00	0.00	0.00	0.03	0.05	0.02	0.06	0.04	0.04
residual Stdev	0.00	47.4%	23.8%	24.9%	20.9%	18.8%	13.9%	19.3%	13.3%	12.4%	13.8%

Table-2 Co-variance Analysis

	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL
SPX	52.7%	63.7%	57.8%	70.8%	60.9%	64.5%	63.0%	41.2%	54.2%	44.0%
NVDA	100.0%	48.7%	52.4%	34.3%	16.0%	33.8%	15.7%	6.0%	16.5%	6.9%
CSCO	48.7%	100.0%	61.4%	48.7%	32.8%	41.0%	29.7%	22.0%	23.9%	16.5%
INTC	52.4%	61.4%	100.0%	41.1%	28.0%	41.2%	28.6%	13.6%	32.5%	11.0%
GS	34.3%	48.7%	41.1%	100.0%	47.2%	49.4%	41.7%	17.3%	29.6%	20.3%
USB	16.0%	32.8%	28.0%	47.2%	100.0%	53.9%	54.0%	33.6%	23.4%	21.8%
TD CN	33.8%	41.0%	41.2%	49.4%	53.9%	100.0%	41.7%	23.1%	27.3%	21.2%
ALL	15.7%	29.7%	28.6%	41.7%	54.0%	41.7%	100.0%	34.6%	45.2%	40.7%
PG	6.0%	22.0%	13.6%	17.3%	33.6%	23.1%	34.6%	100.0%	49.4%	48.3%
JNJ	16.5%	23.9%	32.5%	29.6%	23.4%	27.3%	45.2%	49.4%	100.0%	52.7%
CL	6.9%	16.5%	11.0%	20.3%	21.8%	21.2%	40.7%	48.3%	52.7%	100.0%

MODELS COMPARISON

The Efficient Frontier, Inefficient Frontier, and Minimum Variance Frontier for both the Markowitz Model and Index Model with two separate constraints were calculated using monthly data that was coarse-grained from daily data. We use Solver, an Excel add-in, as the key tool to solve the optimization to compute the three frontiers. We will compare and contrast the findings of Output from MM the two models to determine whether the Index Model is a good enough approximation model for the Markowitz Model.

Comparison of Markowitz Model Constraints vs. IM Model Constraints

To assure portolios with the lowest variation and highest sharo ratio, we might build our investment using excel-solver and the MM and IM assumptions.

Table-3 MM output from Excel-solver

MM (Constr1):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	38.37%	-2.97%	-2.89%	1.33%	-5.90%	-0.30%	19.41%	-11.48%	25.93%	18.83%	19.67%	7.51%	10.95%	0.685	CAL:	0.0%	0.0%
MaxSharpe	-42.74%	15.75%	-1.15%	-6.11%	3.25%	6.48%	35.29%	1.07%	45.71%	30.00%	12.45%	14.01%	13.95%	1.004	2.5	35.0%	34.9%
MM (Constr2):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	38.37%	-2.97%	-2.89%	1.33%	-5.90%	-0.30%	19.41%	-11.48%	25.93%	18.83%	19.67%	7.51%	10.95%	0.685	CAL:	0.0%	0.0%
MaxSharpe	-100.00%	21.50%	0.31%	-8.15%	11.46%	12.25%	44.92%	6.87%	52.33%	41.02%	17.48%	16.56%	16.06%	1.031	2.5	41.4%	40.2%
MM (Constr3):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	38.37%	-2.97%	-2.89%	1.33%	-5.90%	-0.30%	19.41%	-11.48%	25.93%	18.83%	19.67%	7.51%	10.95%	0.685	CAL:	0.0%	0.0%
MaxSharpe	-109.97%	22.46%	0.89%	-8.19%	12.73%	13.21%	46.46%	7.90%	53.50%	42.72%	18.30%	16.99%	16.48%	1.031	2.5	42.5%	41.2%
MM (Constr4):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	9.49%	0.00%	0.00%	0.00%	0.00%	0.00%	19.85%	0.00%	28.91%	20.62%	21.13%	8.88%	11.27%	0.788	CAL:	0.0%	0.0%
MaxSharpe	0.00%	10.95%	0.00%	0.00%	0.00%	0.00%	23.73%	0.00%	42.56%	16.17%	6.60%	12.06%	13.12%	0.919	2.5	30.1%	32.8%
MM (Constr5):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	0.00%	-0.97%	0.08%	2.51%	-0.99%	3.50%	24.70%	-8.17%	28.91%	25.58%	24.85%	8.71%	11.18%	0.779	CAL:	0.0%	0.0%
MaxSharpe	0.00%	14.93%	-6.85%	-10.14%	-1.30%	2.43%	30.65%	-2.27%	43.31%	23.61%	5.64%	13.06%	13.69%	0.954	2.5	32.6%	34.2%

Chart-11

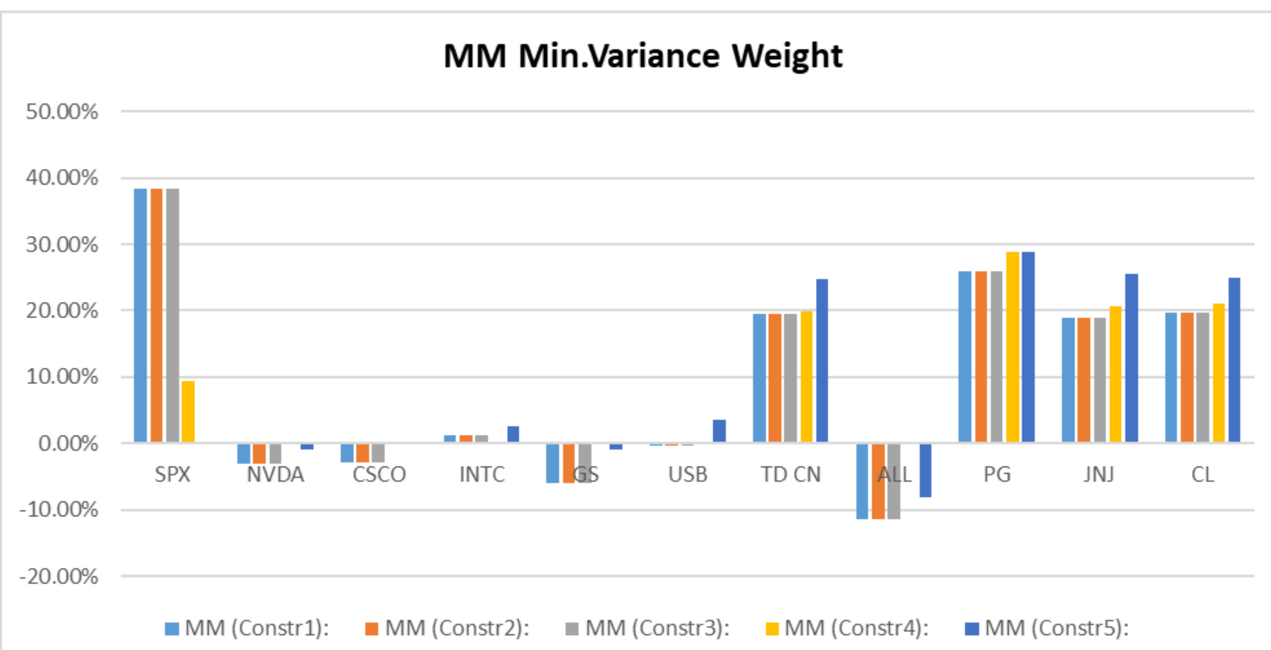
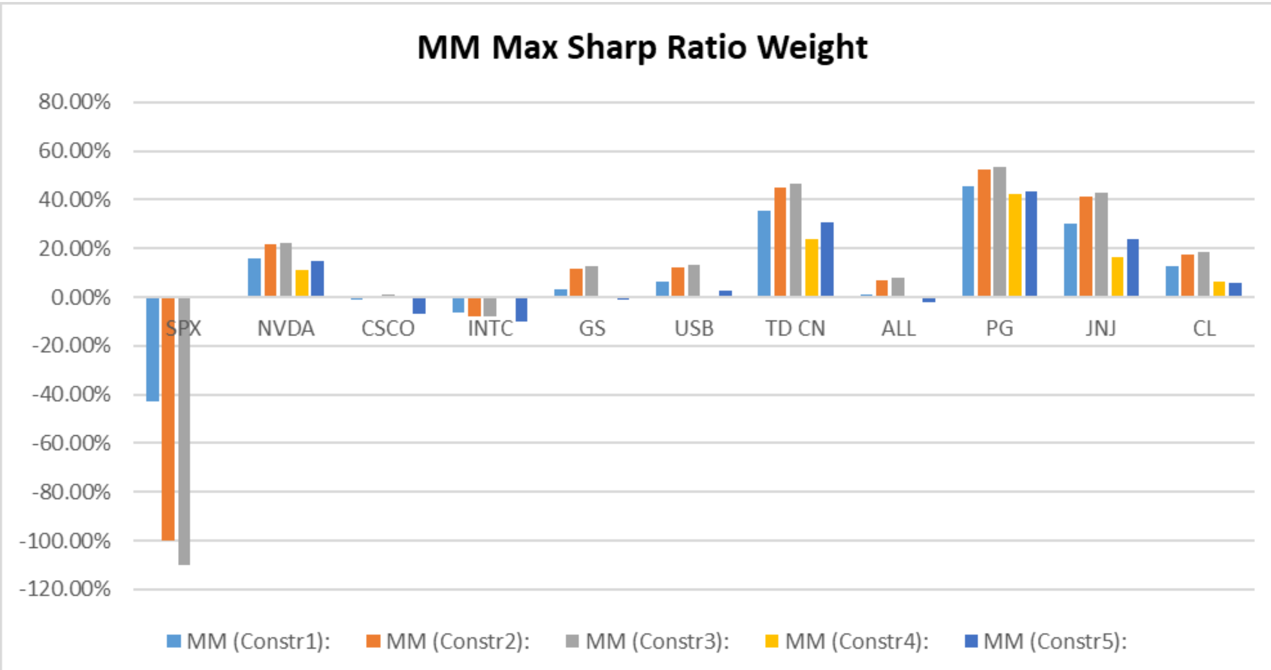


Chart-12



Output from IM

Table-4 IM output from Excel-solver

IM (Constr1):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe		
MinVar	25.62%	-4.04%	-5.27%	-2.81%	-8.73%	0.76%	10.28%	-1.40%	31.27%	27.68%	26.64%	7.15%	9.63%	0.742	CAL:	0.0%	0.0%
MaxSharpe	-47.62%	8.88%	-1.24%	-0.46%	-0.64%	6.67%	26.55%	4.57%	43.66%	33.35%	23.01%	12.07%	12.18%	0.990	2.5	30.2%	30.5%
IM (Constr2):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe		
MinVar	25.62%	-4.04%	-5.27%	-2.81%	-8.73%	0.76%	10.28%	-1.40%	31.27%	27.67%	26.64%	7.15%	9.63%	0.742	CAL:	0.0%	0.0%
MaxSharpe	-70.16%	10.32%	-0.57%	-0.11%	0.57%	6.36%	34.25%	7.35%	46.60%	36.84%	25.11%	12.87%	12.92%	0.996	2.5	32.2%	32.3%
IM (Constr3):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe		
MinVar	25.62%	-4.04%	-5.27%	-2.81%	-8.73%	0.76%	10.28%	-1.40%	31.27%	27.67%	26.64%	7.15%	9.63%	0.742	CAL:	0.0%	0.0%
MaxSharpe	-70.16%	10.32%	-0.57%	-0.11%	0.57%	6.36%	34.25%	7.35%	46.60%	36.84%	25.11%	12.87%	12.92%	0.996	2.5	32.2%	32.3%
IM (Constr4):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe		
MinVar	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	9.21%	0.00%	33.22%	28.88%	28.32%	8.64%	10.16%	0.850	CAL:	0.0%	0.0%
MaxSharpe	0.00%	6.74%	0.00%	0.00%	0.00%	0.00%	17.75%	0.00%	37.34%	22.75%	15.41%	10.71%	11.72%	0.914	2.5	26.8%	26.3%
IM (Constr5):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe		
MinVar	0.00%	-3.32%	-3.28%	-1.14%	-6.02%	3.27%	14.25%	1.13%	34.13%	31.54%	29.47%	7.82%	9.75%	0.802	CAL:	0.0%	0.0%
MaxSharpe	0.00%	7.68%	-5.84%	-4.45%	-6.81%	2.56%	22.82%	0.47%	38.76%	26.66%	18.11%	10.84%	11.48%	0.944	2.5	27.1%	28.7%

Chart-13

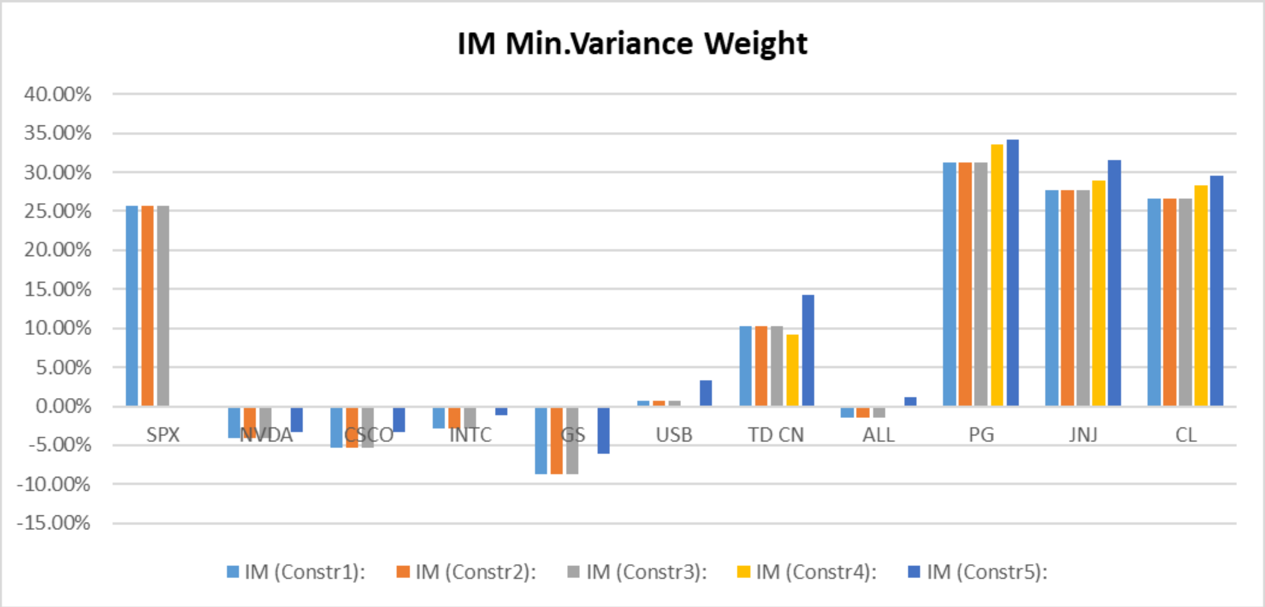
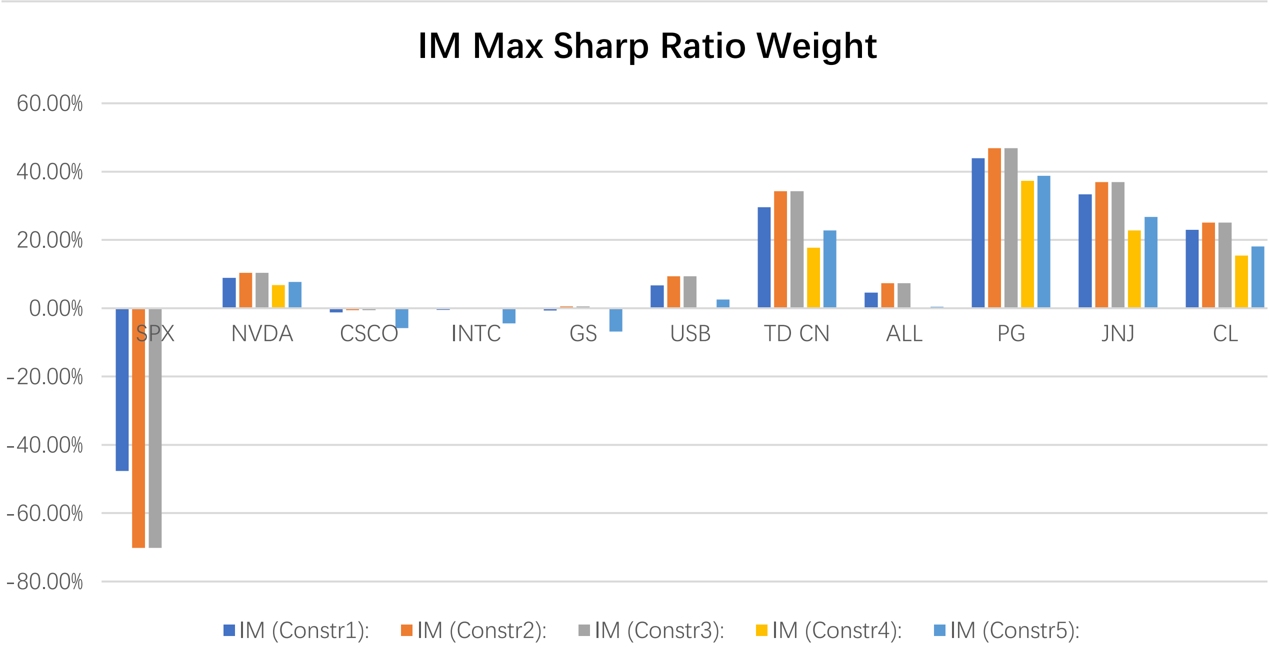


Chart-14



Comparison of Different Weights under Different Models

Min. variance

When the output from calulations, monthly returns, and figures are combined, both models are more inclined to long SPX, TD CN, The Procter & Gamble Company, Johnson & Johnson manufactures healthcare, and Colgate-Palmolive and short NVDA, CSCO, INTC,GS, US Bancorp, and Allstate Corporation due to lower fluctuations under constraint 1,2,3 because of their relative small standard deviation. Interestingly, under constraint 4, mutual funds under MM & IM have a proclivity to choose companies in the pharmaceutical business that have a high risk of securities in the domains of finance and technology. Concerning Constraint-5, the addition of a wide index might short those prominent businesses in finance and high technology in order to reduce risk. To reduce risk, investors that follow MM and IM may choose to short top businesses with excllent 5-year net growth rates and steady P/E ratios in fiance and technology and include benchmark into their portfolio.

However, shorting select riky assets may reduce the risk of the entire portfolio, implying that leverage may be a useful strategy for balancing risk and reward.

Max Sharp ratio

Under permissive constraints, both models are prone to short benchmark with big proportions in order to attain Max.sharp ratio. Furthermore, under MM & IM assumptions, those firms in finance and technology may be better picks for investors.

It can be observed that the IM model's asset allocation position is similar to that of the MM model. However, because the overall allocation ratio of each stock in the two models is similar, the return of the minimum variance portfolio of IM will be lower than the MM model, but its standard deviation is slightly higher, most likely because the MM model as a whole allocates more funds to other stocks, which helps to diversify risks.

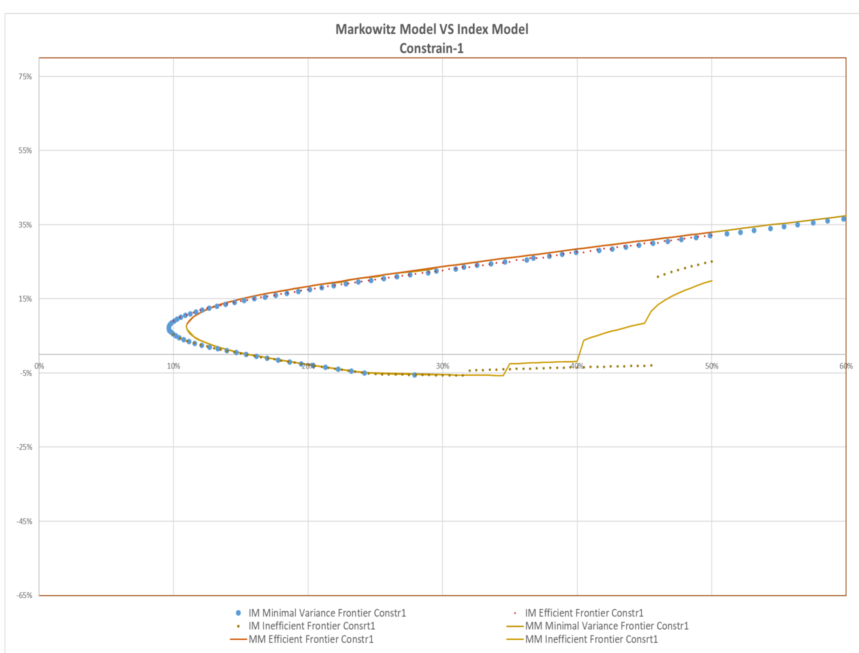
We could buy the lowest-risk investment pool while keeping constraints 1, 2, and 3 in mind in order to get the lowest variance and highest sharp ratio.

To some extent, a portfolio of securities comprised of short positions can help achieve high returns while also controlling risk. Under constraints 4 and 5, investors may modify the weights of securities without changing the proportion of the benchmark in order to obtain the maximum sharp ratio with the least variance.

The risk-free rate of return is denoted as RF, and because its risk (standard deviation) is zero, it indicates that the riskless asset corresponds to a point on the vertical axis. It can be demonstrated that the point Y corresponding to the maximum utility, when the riskless asset F is coupled with the optimal risky portfolio P on the Efficient Frontier, is located between P and F. The CAL line describes all feasible allocations between the ideal risky portfolio and the riskless asset.

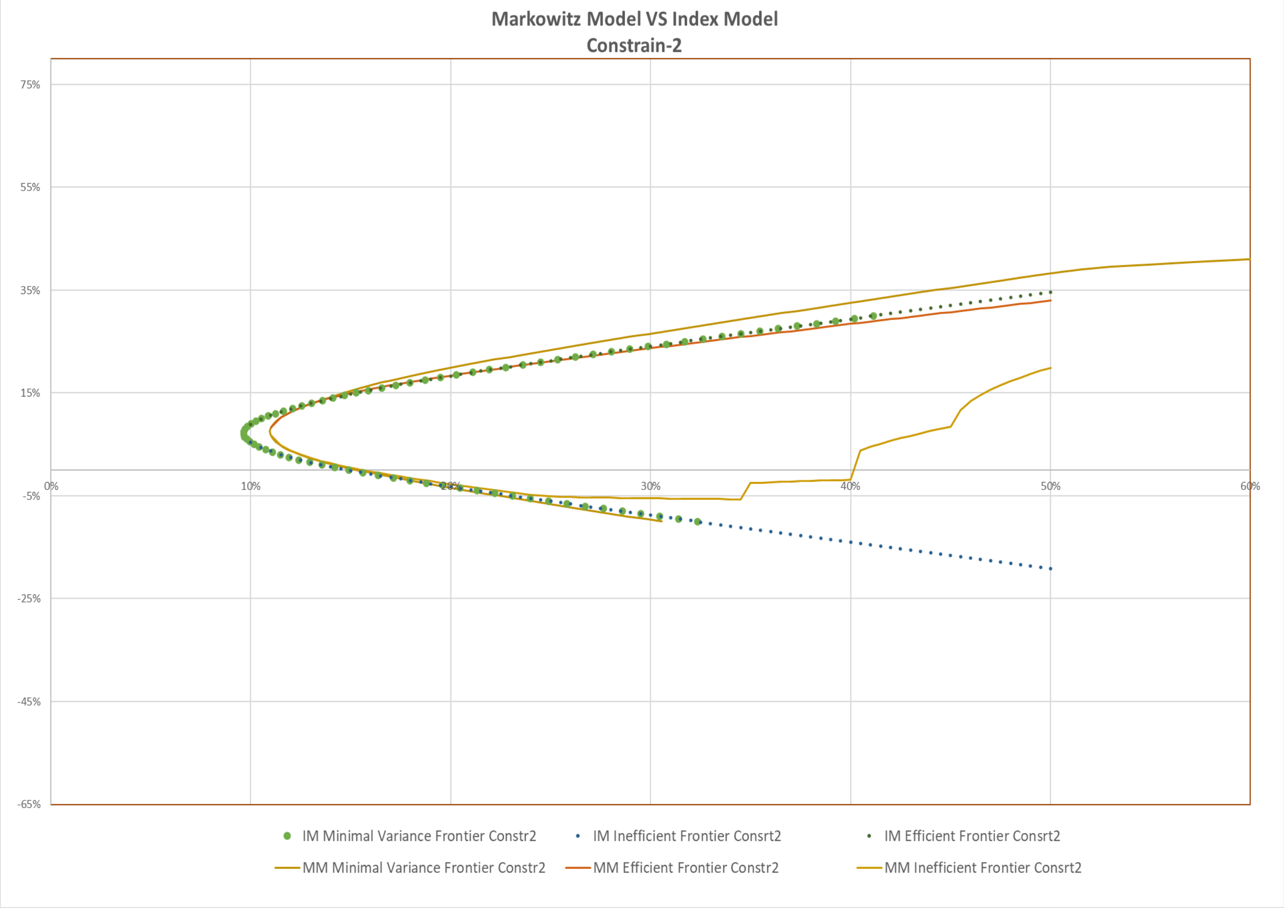
The Sharpe Ratio is the slope of the CAL line, and it represents the risk premium of a portfolio assessed in terms of the risk the portfolio bears. The CAL line depicts the straight line between expected return and risk. The steeper the slope (or the Sharpe Ratio), the greater the projected return for a given degree of risk, or the better the investment. Based on these two outcomes, we may conclude that the Index Model outperforms the Markowitz Model in our particular scenario. In addition to the Markowitz and Index models, there are several more equivalent in spirit multi-factor models that integrate more than one element - the broad index - that are also commonly used in practice. These two models, on the other hand, form the cornerstone of portfolio optimization theory and have a wide range of practical applications.

Chart 15-Comparison of the MM & IM Optimization Problem Solutions under Constrain-1



Constrain-1: This extra optimization constraint is intended to emulate FINRA Regulation T, which permits brokers to allow their customers to have positions that are 50% or more backed by the customer's account equity: $\sum = \leq$ 11 1 2 i wi ; The IM minimum variance frontier, as shown in the graph, can reach the left-most position, which roughly corresponds with the MM Efficient Frontier and the IM Efficient Frontier, allowing the IM model to obtain the same amount of return as MM model did while the standard deviation was rising up.

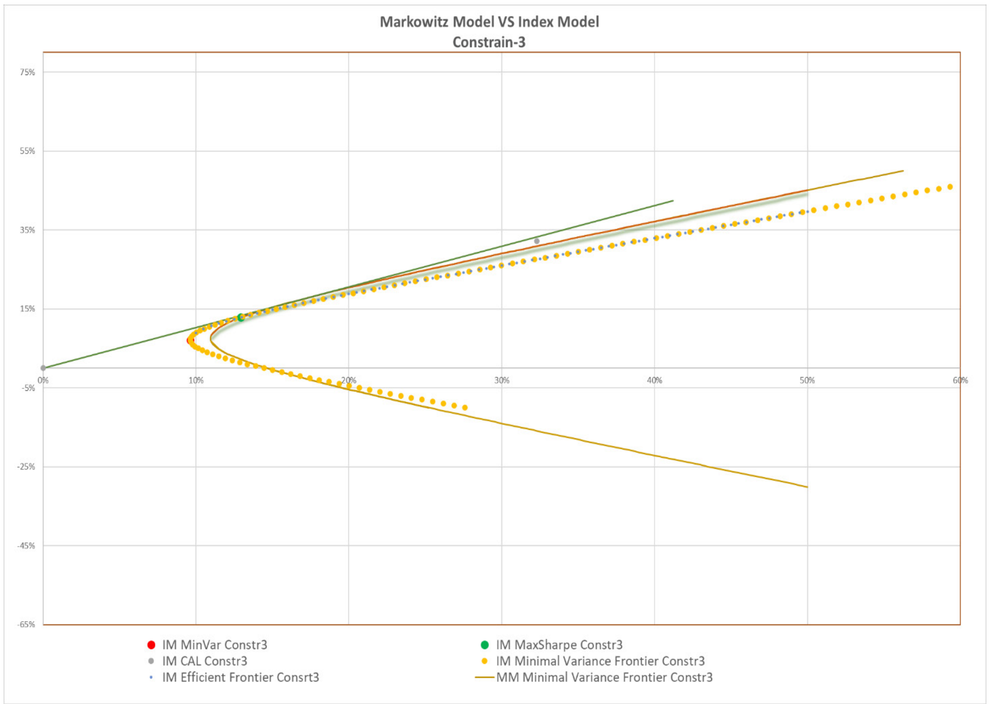
Chart 16-Comparison of the MM & IM Optimization Problem Solutions under Constrain-2



Constrain-2: Investors might earn relative greater returns on the MM model with little variance if constrain-2: simulation of any arbitrary "box" restrictions on all weights of securities in the range of -1 to 1. This new optimization restriction is intended to emulate some arbitrary "box" weight limitations specified by the client: w i i $\leq$,1 for $\forall$;

Due to this constraint, the IM model could still accomplish the risk-minimization combination, but the portfolio under the MM model could produce superior returns when the standard deviation was nearly more than 15%.

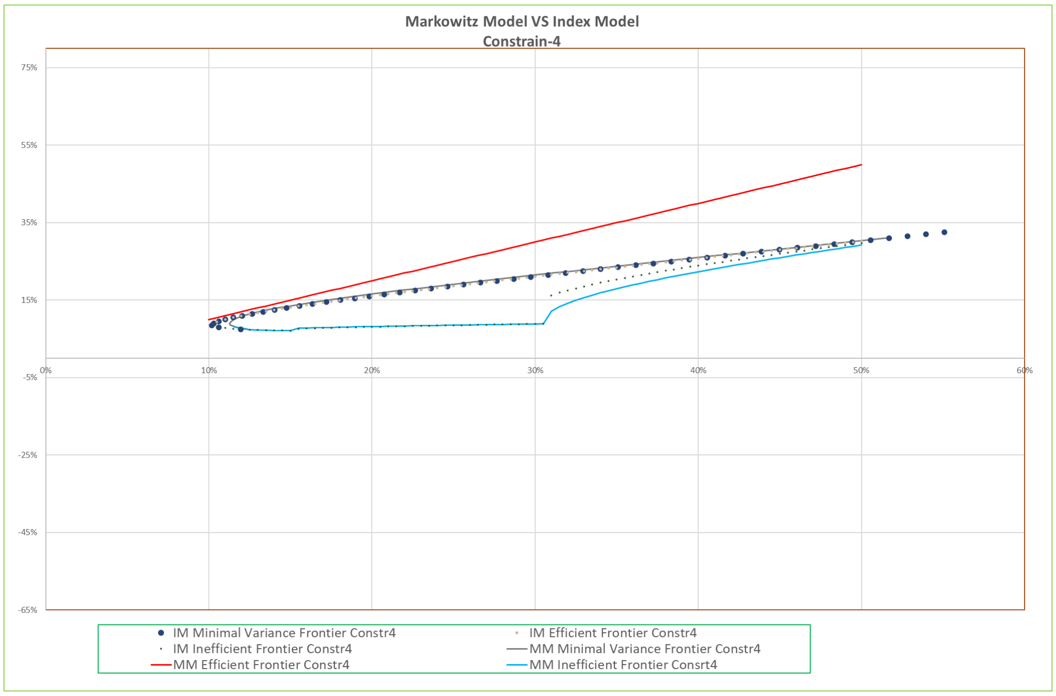
Chart 17- Comparison of the MM & IM Optimization Problem Solutions under Constrain-3



Constrain-3: A "free" issue, with no extra optimization restrictions, to show how the area of permitted portfolios in general, and the efficient frontier in particular, appear when no constraints are applied;

The IM Minimal Variance Frontier still exhibits the lowest risk for the investment pool and is less likely than the MM Minimal Variance Frontier to earn a negative return. The efficient borders under the IM and MM models are apparent to CAL in the absence of additional constraints, and all lines are under CAL.

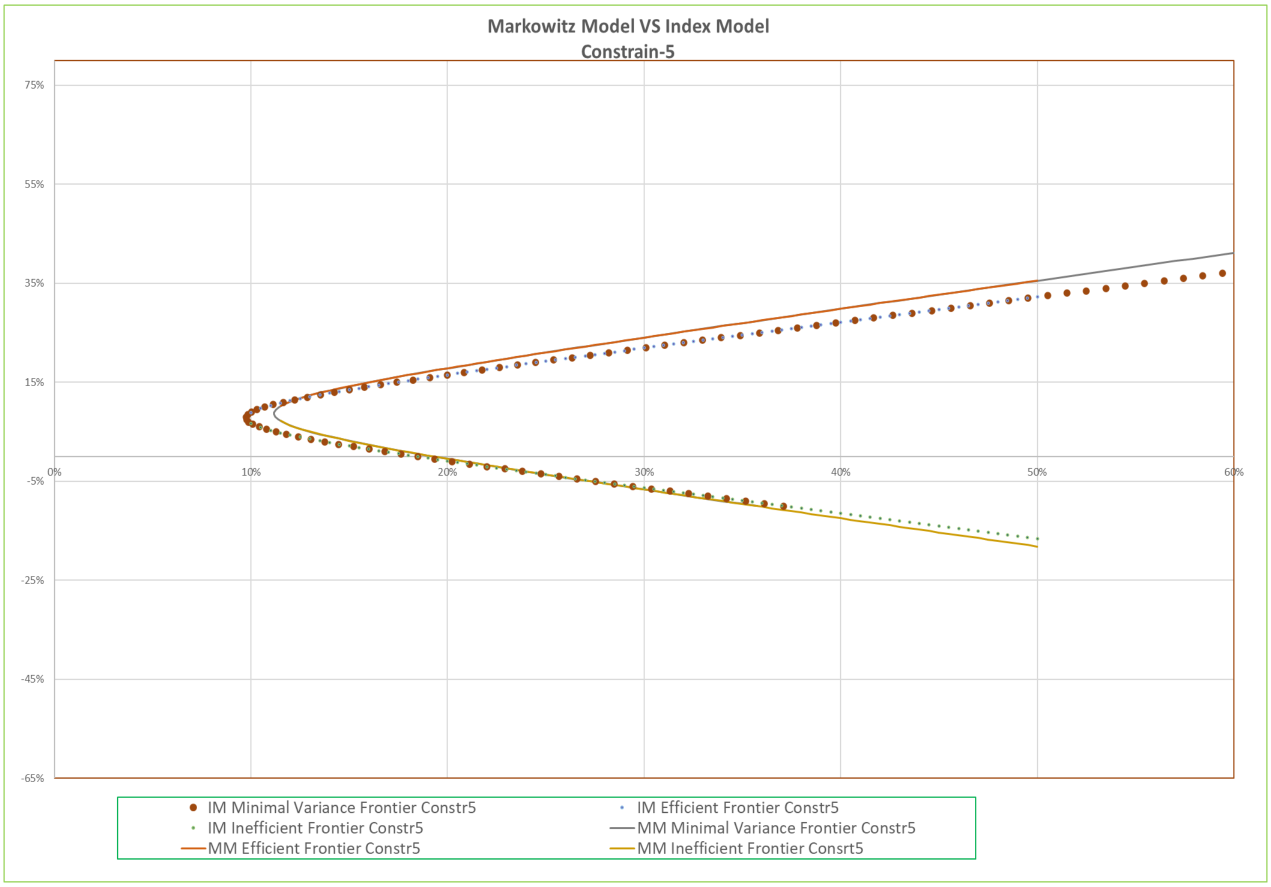
Chart 18-Comparison of the MM & IM Optimization Problem Solutions under Constrain-4



Constrain-4: This new optimization restriction is specifically meant to imitate the typical limits that exist in the US mutual fund industry: a US open-ended mutual fund is not permitted to have any short positions (for more information, see the Investment Company Act of 1940). w i i ≥ ,0 for ∀ ;

A mutual fund with no short strategy can always produce a positive return under the MM & IM model. Furthermore, almost all MM borders overlap with IM boundaries, and when standard deviation exceeds 50%, the inefficient lines may interact with the efficient ones.

Chart 19-Comparison of the MM & IM Optimization Problem Solutions under Constrain-5



Constrain-5: Finally, we want to determine if including the broad index into our portfolio has a beneficial or negative impact, therefore we'll add another optimization constraint :0 w1 = .0

The IM Minimal Variance crosses and walks along the MM Minimal Variance Frontier at roughly 15% by constrain-5. The exposures of the boundaries are rather minor under Constrain-5. Individually, all borders under the MM and IM models are on the same line, yet IM can still win the game with minimum volatility. However, MM was able to obtain a greater sharp ratio and return as the standard deviation increased.

Comparison of Different Constraints for the Index Model

Above analysis depicts a comparison of the Markowitz model and the exponential model under the Constr1 condition. The comparative analysis discovered that, under the condition of Constr1, that is, including the general index SPX, the Markowitz model's minimum variance portfolio and maximum Sharpe ratio portfolio are located in the upper left of the index model, with less risk and higher return, making it a more ideal investment portfolio. Furthermore, the two portfolio points will be closer to the point than the identical model under different restrictions would be.

Overall, the MM model's limits, including the minimal variance boundary, will encircle and encompass more regions than the IM model. For the efficient frontier, the corresponding return of the MM model will be larger in the case of the same standard deviation, and in the case of the same income, the corresponding standard deviation of the MM model will be smaller, indicating that the MM model's efficient frontier will be better. The equivalent return of the MM model for the null boundary will be lower with the same standard deviation, and the standard deviation of the MM model will be less with the same return.

And when the standard deviation grows, the difference between the effective and ineffective frontiers of the two models grows wider and wider. Because the maximum Sharpe ratio of the MM model is greater and the slope is greater, the CAL line will be higher than the IM model. And the difference between the two will be bigger than the difference between the same model under different constraints, since the Sharpe ratio difference between the MM model and the IM model under Constr1 will be greater than the Sharpe ratio difference between the MM model and the IM model under other constraints.

As seen in the above graphics, the Index Model beats the Markowitz Model while attempting to resolve the global min var or max. sharp points using 5 different constrains.

When we look at the CALs, we can see that the Index Model's CAL, whether with any constrains, has a sharper slope than the capital allocation lines in the Markowitz Model. We must incorporate the riskless asset to complete the portfolio.

Markowitz Model under 5 constrains

Chart 20- Data of MM under Constr1-5

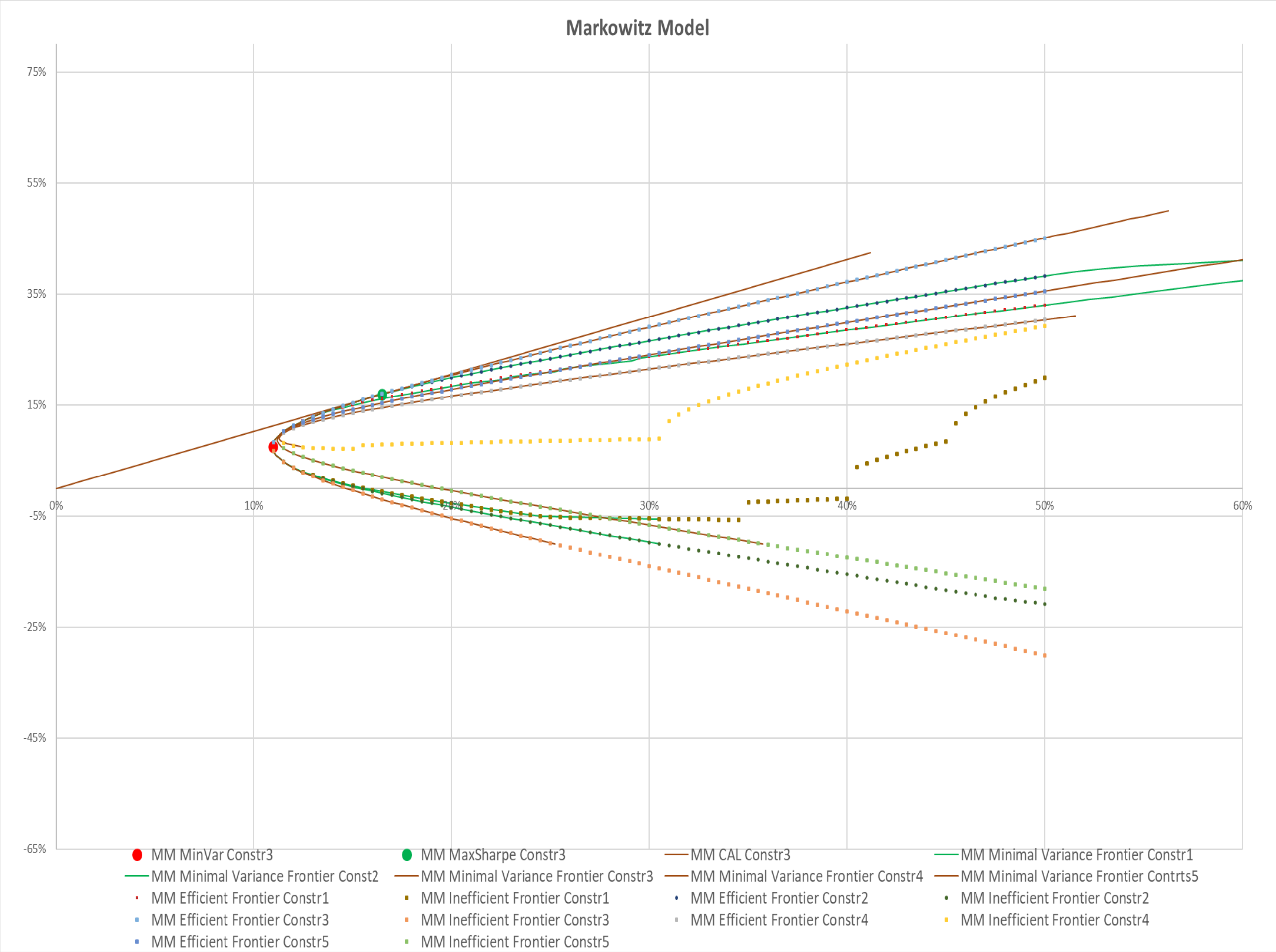
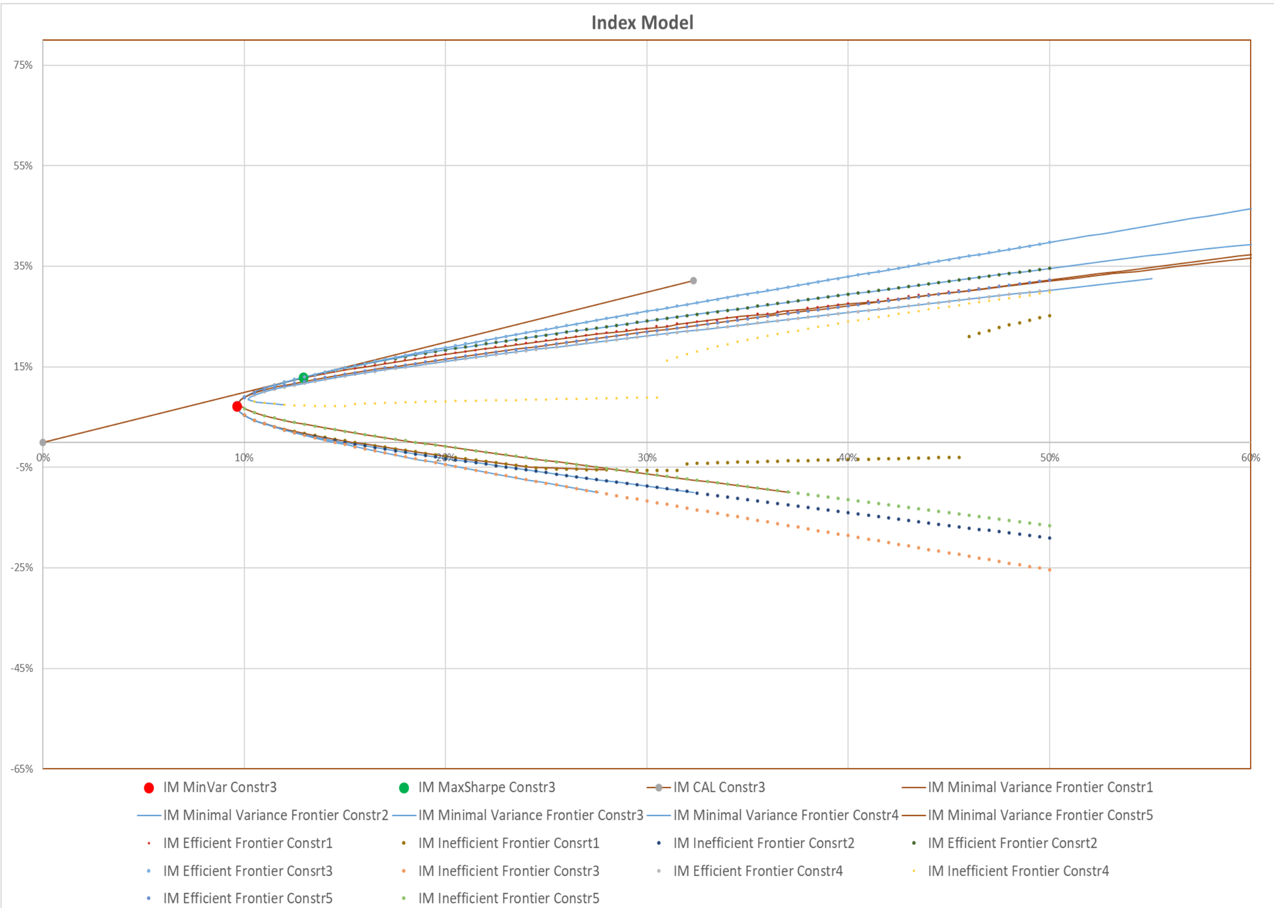


Chart 21- Data of IM under Constr1-5



Analysis of two models under 5 different constrains

Minimum variance: A portfolio with a global minimum risk is one with a global minimum risk. The minimal risk point, as seen in the graphic, is below the CAL line, indicating that investors holding the 10 companies we chose would not be able to benefit from the same returns as those on the CAL. Moreover, both optimization models also show that the global minimal risk portfolio points for securities with restrictions 4 and 5 are higher than risk portfolio points for securities with other limitations. Only lines under limitation 4 were not able to reach the left-most point for variance reduction of portfolio. However, the picture still demonstrates that mutual funds or investors without a short position in hazardous assets are unable to achieve the least variance of investment decisions.

MAX. sharp ratio A Sharpe Ratio describes asset performance in terms of portfolio risk. The performance of a portfolio is assessed by comparing it to the rate of return on existing risk-free investments. A greater maximum Sharpe Ratio suggests that the assets in the portfolio will outperform.

As a result, the portfolio with restriction 3 has the highest maximum Sharpe Ratio, implying that its performance is superior to the portfolio with other constraints. However, investors cannot get a higher sharp ratio by imposing constraints 4&5, and more particularly, selecting stocks without any extra constraints (under constrain-3) might, to the best degree possible, create an ideal portfolio with a good balance of risk and return. Finally, the Capital Allocation Line without any regulations has the largest slope than the Capital Allocation Line for other constraints.

This indicates that an investor using the free method receives a higher projected return for incurring the same risk as an investor utilizing other guidelines. Due to the limited degree of leverage, the mutual fund could only achieve the minimal maximum sharp ratio without a short position.

The portfolio with limitation 3 has a substantially greater average return than the portfolio with the other limits for the same level of risk, according to the Markowitz Model. With the known risks connected with the portfolio, choosing stocks at random delivers the maximum predicted return, as shown by the standard deviation, which measures risk. Intriguingly, the Markowitz Model inefficient frontier for constraint 4 with continuous positive return shows a larger maximum sharp ratio, and investors may also see the maximum sharp ratio if the portfolio contains more than 50% of the customer's equity account.

The projected returns for the inefficient frontier curve with constraint 3 are much lower than those for the inefficient frontier curve with others. The inefficient frontier curve depicts a portfolio of investments that does not produce returns commensurate with the volume of risks involved. Regarding further constraints, three boundaries are downward and all are below the line with restriction number 4.

Conclusion

The IM optimization model improves on the MM optimization model in that it requires fewer estimations to determine portfolio risks and returns. Using the IM and MM, this study examines the regions of permissible portfolios under various restrictions. The Markowitz Model's efficient frontier for constraint 3 was higher than that of the other limitations. Constraint 3 was less stringent for the inefficient frontier curve. With a low variance frontier, a high Sharpe ratio, a low capital allocation line, and a reduced risk portfolio, Constraint 3 was also more severe. The IM evaluates restricted data. The frontier curve for constraint number 3 is more effective than the others. In comparison to previous restrictions, restriction 3's minimal risk portfolio ratio was smaller. The outcomes for the highest Sharpe Ratio were similar.

It is evident from a comparison of the two optimization models' results that IM is more effective than MM. Despite comparable outputs, the IM produces superior results. The values of the efficient frontier curves of IM are lower. The expected returns for a specific amount of risk are displayed on the efficient frontier curves. For constraint 1, the maximum Sharpe Ratio values were higher than they were for the other constraints. IM shows that constraint 3 yields the most revenues while posing the fewest risks. In a same vein, IM offered a higher maximum Sharpe Ratio than others for constraint 3. In contrast, the maximum Sharpe ratio for MM was higher overall than it was for IM. The portfolio's assets fared well, according to the MM's results about the Sharpe ratio. On the other hand, the negative maximum Sharpe Ratio from the IM shows that the portfolio group's equities did not perform as expected. These results might be attributable to the portfolio holding equities with poor returns or to risk affecting predicted returns (Shadabfar & Cheng, 2020). On the Capital Allocation Line, Constraint 3 has a superb slope from both the MM and IM. In contrast, the IM produced a slope that was steeper than the MM. Both risky and low-risk goods are part of the capital allocation line in a portfolio. It illustrates the possible returns for investors who are willing to take some risks. As a consequence, in terms of returns from assets with defined risk, the IM beat the MM. In general, investors should feel free to mix long and short positions as part of their investing strategy when choosing stocks from high-risk businesses or sectors like banking or technology in order to optimize anticipated return while minimizing risk while maintaining consistent expected return. Thus, the efficient portfolio shows the weights of the securities that, according to the MM or IM, offer the highest returns for a given level of risk or the lowest risks for a given level of expected return, respectively. Additionally, investors with large portfolios like the IM model while those with smaller portfolios can opt for the MM model. The IM model is therefore preferred by investors with bigger portfolios, but the MM model may be preferred by investors with smaller portfolios.

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