Portfolio Optimization and Analysis Using Modern Portfolio Theory

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Abstract:

In investment transactions, such as stocks and commodities, risk is always involved. The link between investment returns and risk factors is often discussed. Many academics have attempted to develop models under any expected rate of return. The primary purpose of this paper is to demonstrate the application of modern portfolio theory in optimizing investment portfolios. The paper mainly analyzes the viewpoint through the use of historical financial data. The study constructs and examines portfolios using the Full Markowitz Model (MM) and the Index Model (IM) through five constraint conditions. By incorporating various constraints, the study aims to understand how regulatory, industry-specific, and client-driven limitations impact portfolio construction and performance.

Keywords: Investment; Risk management; Modern Portfolio Theory

Introduction

When investing or trading stocks and commodities, two primary considerations must always be kept in mind - potential returns on investment and risk. Arratia (14) established one fundamental principle, the No Arbitrage Principle, which demonstrates that there can never be a free lunch in financial markets. Money cannot simply appear out of thin air since it requires initial investments that require risk in exchange for returns. This principle implies that gains cannot be realized without taking on some associated risk. The relationship between risk and return becomes particularly evident when working with securities situated along an efficient frontier. Pursuing a perfect investment that combines high returns with no risk is nearly impossible to obtain, as Elton and Gruber (13) assert. Thus, individuals have spent years devising methods and theories that approximate this ideal state. However, none of the planned methods are as famous and influential as Modern Portfolio Theory (MPT). Developed by Harry Markowitz in 1952, MPT revolutionized portfolio management by introducing a systematic approach to achieve this balance between returns and risks (Fabozzi et al. 20). MPT recognizes that investors are not solely concerned with maximizing returns; they also seek to minimize risk. This theory emphasizes the importance of diversification and efficient asset allocation to achieve the best risk-adjusted returns. Since then, numerous scholars have attempted to develop improved models with minimal variance under any expected rate of return. Sharpe proposed his Single Index Model (SIM) based on Markowitz's original framework (Mandal 19). SIM was seen as attractive because its calculation process greatly simplified the Markowitz model. It created an easy framework for portfolio optimization that still enjoys widespread popularity worldwide among financial practitioners and individual investors.

The primary objective of this project is to practically demonstrate the application of Modern Portfolio Theory in optimizing investment portfolios. Using historical financial data, the study constructs and analyzes portfolios using the Full Markowitz Model (MM) and the Index Model (IM) through five constraint conditions. By incorporating various constraints, the study aims to gain insights into how regulatory, industry-specific, and client-driven limitations impact portfolio construction and performance.

Data Collection and Preparation

The foundation of any data-driven project is the dataset itself. The dataset in the present study spans a period of twenty years, from May 11, 2001, to May 12, 2021. It includes daily total return data for eleven assets, encompassing the S&P 500 index and ten individual stocks, as shown in Table 1 below. Additionally, the paper extracts the 1-month Fed Funds rate as a proxy for a risk-free rate. These assets were chosen to represent diverse sectors, enabling the study to gain comprehensive insights into their roles in portfolio optimization. The historical daily total return and S&P 500 index data for each asset were collected from reliable financial sources (Yahoo Finance), ensuring accuracy and reliability.

Table 1. Study Stocks Selection

#	Group #4	Full Name	Sector (Yahoo finance)
1	QCOM	QUALCOMM Incorporated	Technology
2	AKAM	Akamai Technologies, Inc.	Technology
3	ORCL	Oracle Corporation	Technology
4	MSFT	Microsoft Corporation	Technology
5	CVX	Chevron Corporation	Energy
6	XOM	Exon Mobil Corporation	Energy
7	IMO	IMPERIAL OIL LIMITED	Energy
8	KO	The Coca-Cola Company	Consumer Defensive
9	PEP	PepsiCo, Inc.	Consumer Defensive
10	MCD	McDonald's Corporation	Consumer Cyclical

To align with the principles of Modern Portfolio Theory, which primarily operates every month, the raw daily data is aggregated into monthly observations. This aggregation process involved calculating the total return for each month. The approach simplifies calculations and aligns study data with the monthly optimization models employed in the analysis. Before conducting portfolio optimization, it was necessary to transform the raw total return data into essential parameters. These parameters include asset returns, volatility, and correlations. Asset returns were calculated by aggregating the monthly full recoveries, and volatility was computed as the standard deviation of returns. Additionally, the correlations between asset pairs were computed to quantify their relationships.

Optimization and Portfolio Construction

Calc*ulation of Optimization Inputs*

The foundation of robust portfolio management rests upon the meticulous calculation of optimization inputs, which are the bedrock of crafting portfolios that strike the ideal equilibrium between risk and return. This study's inputs comprise the mean returns of individual assets, the correlation matrix reflecting their intricate relationships, and the risk-free rate, each of which is fundamental to Modern Portfolio Theory (MPT) tenets that form the crux of the optimization process. After calculating the stock price, the study thoroughly analyzed several related indicators: annualized average return, annualized StDev, beta, annualized alpha, and annualized residual StDev.

Table 2. Various Data Analyses for ten stocks

	SPX	QCOM	AKAM	ORCL	MSFT	cVX	XOM	IMO	KO	PEP	MCD
Annual Average Return	7.5%	13.1%	28.1%	11.1%	13.1%	8.8%	5.4%	10.9%	7.0%	7.9%	13.5%
AnnualStDev	14.9%	33.3%	63.1%	27.8%	23.3%	22.3%	20.8%	30.5%	16.3%	15.1%	18.7%
beta	1.00	1.25	1.65	1.02	1.00	0.92	0.79	1.07	0.54	0.53	0.68
alpha	0.00	0.04	0.16	0.03	0.06	0.02	-0.01	0.03	0.03	0.04	0.08
residual Stdev	0.0%	27.6%	58.1%	23.3%	17.9%	17.6%	17.1%	26.0%	14.2%	12.9%	15.7%

Table 1 shows the descriptive statistics of selected stocks. The results reveal that the 'AKAM' has the highest average rate of return and the most significant standard deviation. 'XOM' 's average return ranks the lowest, while 'PEP' has the most minor standard deviation among the 10 sample stocks. Moreover, Table 3 illustrates the correlation strength between pairs of companies, accentuating the degree of association. Significantly, the table prominently emphasizes both the maximum and minimum correlation values, thereby augmenting the clarity of the presentation.

Table 3. Correlation Coefficient of 10 Companies

	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD
SPX	100.0%	55.7%	38.9%	54.6%	63.9%	61.3%	56.8%	52.2%	49.1%	52.2%	53.7%
QCOM	55.7%	100.0%	27.8%	28.5%	37.5%	23.3%	23.5%	27.2%	19.7%	26.3%	26.2%
AKAM	38.9%	27.8%	100.0%	24.2%	25.6%	12.2%	6.8%	12.7%	8.5%	10.2%	29.1%
ORCL	54.6%	28.5%	24.2%	100.0%	47.5%	26.4%	30.1%	23.3%	6.8%	20.5%	13.7%
MSFT	63.9%	37.5%	25.6%	47.5%	100.0%	33.9%	30.4%	25.0%	27.9%	33.4%	35.7%
CVX	61.3%	23.3%	12.2%	26.4%	33.9%	100.0%	82.9%	73.4%	40.2%	27.2%	39.4%
XOM	56.8%	23.5%	6.8%	30.1%	30.4%	82.9%	100.0%	69.7%	33.8%	24.0%	34.0%
IMO	52.2%	27.2%	12.7%	23.3%	25.0%	73.4%	69.7%	100.0%	29.7%	17.8%	26.8%
KO	49.1%	19.7%	8.5%	6.8%	27.9%	40.2%	33.8%	29.7%	100.0%	57.9%	49.9%
PEP	52.2%	26.3%	10.2%	20.5%	33.4%	27.2%	24.0%	17.8%	57.9%	100.0%	47.0%
MCD	53.7%	26.2%	29.1%	13.7%	35.7%	39.4%	34.0%	26.8%	49.9%	47.0%	100.0%

Opt*imization Models*

Portfolio Theory offers solutions for adequate diversification by allocating investments across assets in an uncertain environment to spread risk and maximize efficiency effectively. One powerful method used in Portfolio Theory is The Markowitz Model (MM). The MM is often called an efficient frontier of risky assets. In contrast, Sharpe's Single Index Model (SIM) simplifies efficient frontier calculation by considering only market factor correlation calculations as part of efficient frontier calculations. As part of an optimal portfolio construction, multiple steps should be followed. Firstly, the risk-return combinations must be calculated, and the minimum-variance frontier plotted (represented by the blue curve in Figure 1) to depict the lowest possible variance given an expected return. Secondly, the global minimum-variance portfolio G is computed. It holds the most minor standard deviation among available portfolios. The efficient frontier lies above this point (See Figure 1).

The third step involves finding an optimal risky portfolio by seeking the steepest Capital Allocation Line (CAL). Steeper CALs bring greater rewards from bearing risk, leading to higher Sharpe ratios (Damodaran 17). In this scenario, Portfolio P is identified by CAL (P). When capital allocation occurs at this step level, more risk-averse investors tend to allocate more assets into risk-free accounts while allocating less to Portfolio P. This results in what is referred to as separation properties.

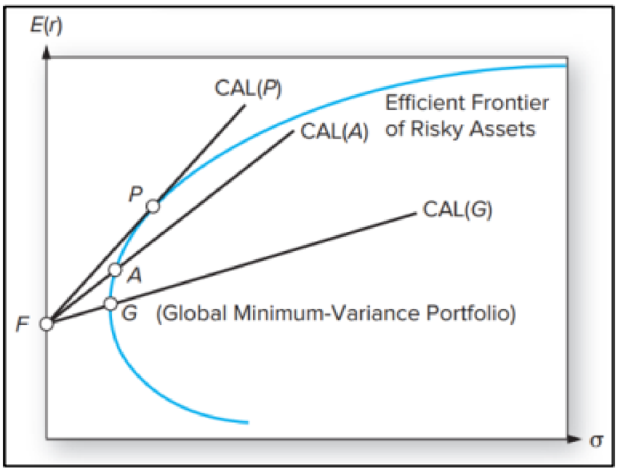

Fig. 1 The efficient set of portfolios

The fundamental formula of the Markowitz Model can be summarized as follows:

Minimize $\sigma^2(rp) = \sum \sum \omega i * \omega j * cov(ri, rj)$

Subject to: $\sum \omega i = 1$ and $rp = \sum \omega i * ri$

Where:

rp is the portfolio's rate of return,

ri is the rate of return of the i-th security,

ωi represents the weight of the i-th security,

$\sigma^2(rp)$ is the variance of the portfolio, and

$cov(ri, rj)$ denotes the covariance between ri and rj.

The formula for the Single Index Model is presented as follows:

Minimize $\sigma^2(rp) = \sum \sum \omega i * \omega j * \beta i * \beta j * \sigma^2(M) + \sum \omega i * \sigma^2(ei)$

Subject to: $\sum \omega i = 1$, $\sum \omega i * ri - \beta i = \beta p$, and $\sum \omega i * \alpha i + \beta p * RM = Rp$

Where:

M denotes the common macroeconomic factor or the market index,

Rp is the portfolio's excess return,
RM is the excess return of the market index,
$\sigma^2(M)$ is the variance of the market index,
ωi represents the weight of the i-th security,
αi is the expected return of the i-th security under neutral market conditions ($RM = 0$),
βi reflects the security's responsiveness to market movements and
ei represents the residual or unexpected return attributed to firm-specific uncertainty.

The Additional Constraints

To unlock diverse scenarios, five additional optimization constraints were applied to the MM and IM models:

Case 1: Regulation T by FINRA Constraint

The Regulation T constraint, inspired by the guidelines set forth by the Financial Industry Regulatory Authority (FINRA), introduces a practical limitation on portfolio composition. This constraint reflects that broker-dealers must ensure customers' positions are funded by at least 50% of their equity. This limitation prevents excessive leveraging and aims to protect both investors and the stability of financial markets.

Mathematical Expression:

$\Sigma wi \leq 0.5$, for all i = 1 to 11

Case 2: Arbitrary "Box" Constraints on Weights

In many investment scenarios, clients express preferences for specific asset allocations. The arbitrary "box" constraints allow us to simulate scenarios where clients impose weight restrictions on individual assets. These constraints reflect investor preferences, industry sector considerations, or ethical considerations that may influence the composition of a portfolio.

Mathematical Expression:

$wi \leq 1$, for all i = 1 to 11

Case 3: No Constraints: A Free Problem

To understand the impact of constraints, the study also considers a scenario with no additional restrictions. This "free" problem is a reference point, allowing comparison portfolio optimization outcomes when no limitations are imposed.

Case 4: U.S. Mutual Fund Industry Limitations

The U.S. mutual fund industry has specific regulations that impact portfolio composition. Mutual funds are prohibited from holding short positions. This constraint is incorporated to emulate the limitations faced by mutual fund managers, focusing on constructing portfolios that exclude short positions.

Mathematical Expression:

$wi \geq 0$, for all i = 1 to 11

Case 5: Impact of Including the Broad Index

Lastly, the study considers the influence of including the broad market index (S&P 500) in the portfolio. By constraining the index's weight to be non-negative, explore whether incorporating the index positively or negatively affects portfolio performance.

Mathematical Expression: $w1 = 0$

Results and Analysis

This paper analyzes the outcomes from MM and SIM to prove their efficacy and then compares their outputs in both models to highlight the differences in how the models are applied to these stocks. In addition, the paper examines the results under similar conditions for different models to determine which performs better. For each of the graphs, the x-axis represents the standard deviation, which is annualized, which indicates the risk level, while the y-axis represents how much annualized returns are.

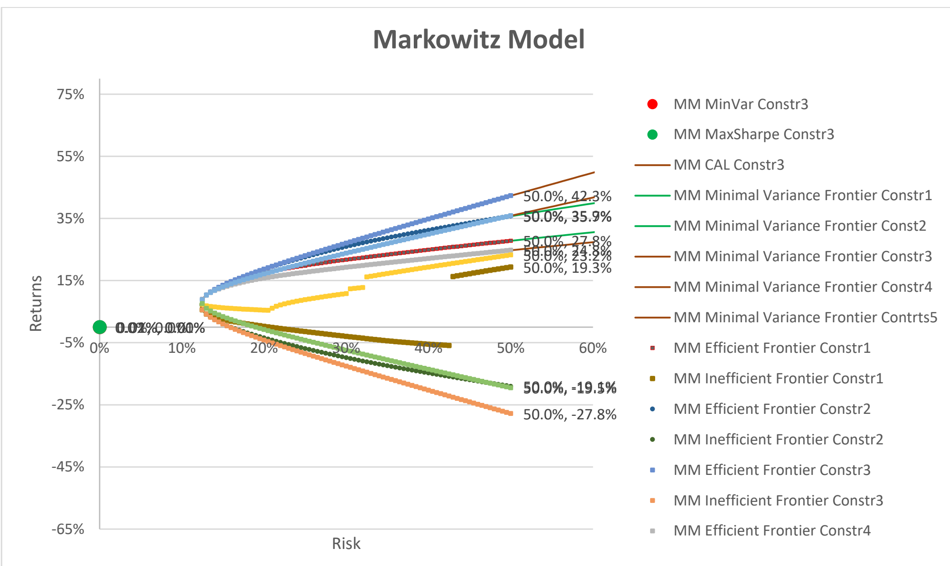

Fig. 2. The outputs of the Markowitz Model (MM)

Figure 3 illustrates the minimal variation frontier and the efficient frontier that is not subject to additional optimization constraints, as well as those under the control of the Markowitz model. The constrained minimal variance frontier is considerably reduced compared to the unconstrained situation. For example, with the same risk, the maximum return and the risk of loss for the constrained portfolio are tiny, and the highest loss is reduced. This suggests that the constraints protect individuals with low-risk tolerance. On the other hand, Figure 4 shows the results of the Index model and that the minimum-variance frontier is constrained, and the efficient frontier is narrowed along with the inefficient border.

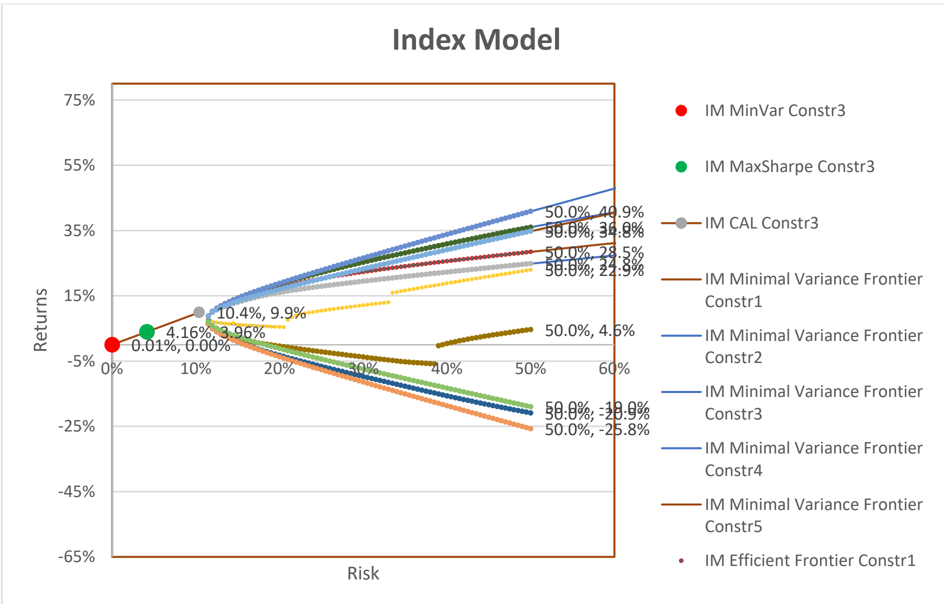

Fig. 3. The outputs of the index model (IM)

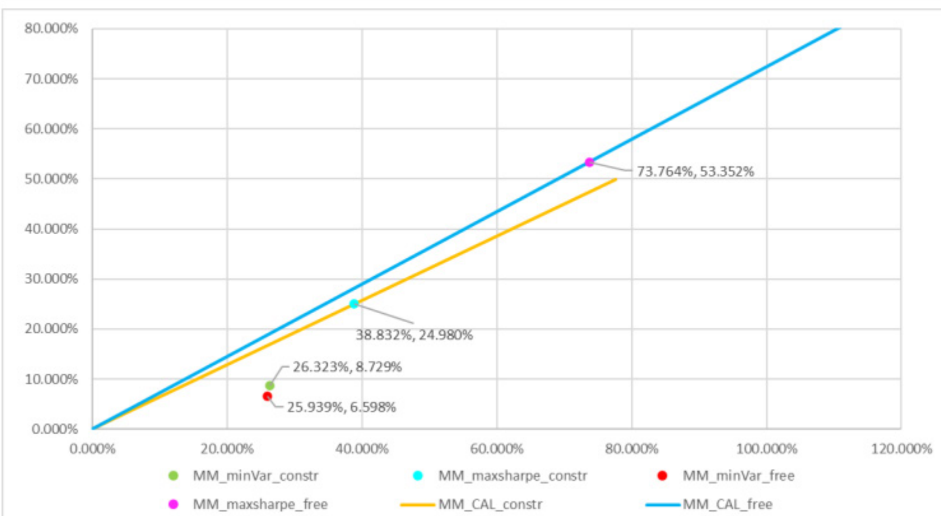

Fig. 4. Comparison of the MM outputs under unconstrained and constrained conditions

Figure 4 compares the capital allocation line, the optimal portfolio, and the minimal variance portfolio in both constrained and free-from-constraint conditions in the Markowitz Model. The blue line shows the CAL not subject to additional constraints, and the orange line shows the CAL with other rules. The blue line slope is higher than the orange line, meaning that, under the same risk levels, the return for controls is lower than the return without extra restriction. The abscissa value of the purple line is 73.764 percent, which is the average deviation of the optimal portfolio that MM can create with no additional conditions. The coordinate of the dot in purple is 53.352%, which indicates the portfolio's returns. The bright blue dots indicate the maximum Sharpe ratio portfolio with additional optimization constraints, where the standard deviation is 38.832 percent, and the return is 24.980 percent. The limitation of not allowing short sales reduces the risk by about 35% but at the expense of a loss of nearly 30 percent yield. For the portfolio with minimum variance, the coordinates of the green and red dots do not differ much; this suggests that the restriction is not a significant factor for investors pursuing a minimum-variance portfolio.

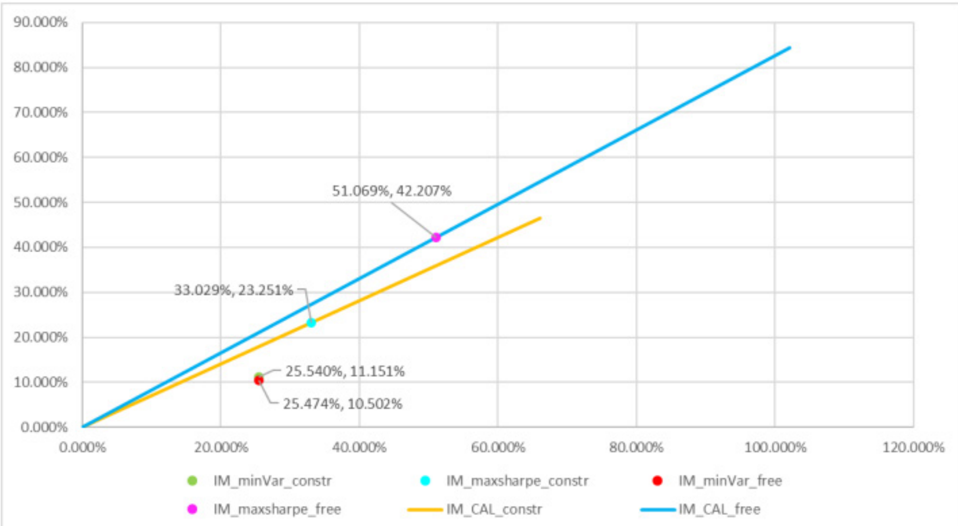

Fig. 5. Comparison of the SIM outputs under unconstrained and constrained conditions

Figure 5 shows the capital allocation line in an optimal portfolio and the minimal-variance portfolio in constraints and not-constrained conditions of SIM. The conclusion derived from the graph is identical to that of Figure 4. However, there are some slight distinctions. First, the two minimum-variance portfolios with constraints or no restrictions using one index nearly match. In addition, the variance between the two portfolios is less than the difference between Markowitz models.

In addition, as illustrated in Figure 2, the lower and upper portions of the two lines are almost identical. However, the part over the minimum-variance portfolios of the SIM is always a better return than MM for the same risk. Additionally, the natural curves of blue and orange are the most efficient borders between SIM and MM and MM, respectively. Based on that, the standard deviation is identical; the blue curve's coordinate is always higher than the orange curve's. In addition, the red and blue lines constitute the CALs for SIM and MM. The slope of the blue line is more than that of the red line. In the past, higher CAL means higher returns for carrying any risk.

The abscissa and the coordinate of the point in purple are 33 percent and 23 percent, respectively. This means that the Sharpe ratio to determine the most optimal portfolio for SIM can be calculated as 0.75. The abscissa in Green Point is 38 percent, and the ordinate is 25 percent. The Sharpe ratio for an optimal portfolio for MM is 0.6. Therefore, the optimal portfolio of SIM is sharper. Then, the red dot on this figure is the minimum variance portfolio of SIM. The abscissa and the red dot ordinate are 25 percent and 11 percent, respectively. The black dot corresponds to the portfolio with the lowest variance of the MM with coordinates of 26 percent and 8 percent, respectively. When you look at the efficiency frontier, the frontier of minimum conflict, and the minimum-variance portfolio, the optimal portfolio and the CAL outcomes generated from the SIM are superior to the MM. This proves that the model with a single index can perform better in this market.

After calculating and observing the performance of the Markowitz and Index models, respectively, this paper has some observations from Table 4 and Table 5.

Table 4. Markowitz Model

MM (Constr1):	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MinVar	30.95%	-2.19%	-1.05%	5.33%	-0.38%	-8.70%	19.56%	-3.86%	20.96%	30.52%	8.86%	7.23%	12.28%	0.589
MaxSharpe	-40.93%	6.04%	6.23%	14.92%	20.88%	5.77%	-9.07%	11.51%	5.12%	28.98%	50.55%	14.59%	16.11%	0.905
MM (Constr2):	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe

MinVar	30.95%	-2.19%	-1.05%	5.33%	-0.38%	-8.70%	19.56%	-3.86%	20.96%	30.52%	8.86%	7.23%	12.28%	0.589
MaxSharpe	-100.00%	12.23%	8.40%	24.18%	30.41%	26.11%	-30.03%	18.09%	9.45%	38.51%	62.65%	17.90%	19.19%	0.933
MM (Constr3):	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MinVar	30.95%	-2.19%	-1.05%	5.33%	-0.38%	-8.70%	19.56%	-3.86%	20.96%	30.52%	8.86%	7.23%	12.28%	0.589
MaxSharpe	-105.17%	12.75%	8.63%	24.86%	31.28%	26.98%	-30.47%	18.51%	9.72%	39.23%	63.68%	18.15%	19.45%	0.933
MM (Constr4):	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MinVar	19.75%	0.00%	0.00%	6.28%	0.00%	0.00%	11.18%	0.00%	21.00%	33.46%	8.34%	8.03%	12.40%	0.647
MaxSharpe	0.00%	2.11%	5.34%	8.38%	14.83%	0.00%	0.00%	5.30%	0.00%	19.71%	44.35%	12.76%	15.06%	0.847
MM (Constr5):	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MinVar	0.00%	0.81%	-0.07%	9.18%	3.72%	-5.11%	21.75%	-2.79%	24.81%	36.09%	11.60%	8.16%	12.46%	0.655
MaxSharpe	0.00%	2.41%	4.82%	11.57%	16.17%	13.02%	-32.66%	13.30%	-0.89%	21.69%	50.59%	14.38%	16.54%	0.870

Table 5. Index Model

MM (Constr1):	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MinVar	12.91%	-4.22%	-2.50%	-0.52%	-0.09%	3.30%	9.10%	-1.38%	29.76%	36.72%	16.92%	7.54%	11.36%	0.664
MaxSharpe	-48.43%	4.39%	6.20%	6.53%	21.74%	4.44%	-1.56%	3.81%	20.48%	33.53%	48.87%	13.63%	14.55%	0.937
MM (Constr2):	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MinVar	12.91%	-4.22%	-2.50%	-0.52%	-0.09%	3.30%	9.10%	-1.38%	29.76%	36.72%	16.92%	7.54%	11.36%	0.664
MaxSharpe	-93.84%	8.18%	7.93%	10.73%	29.77%	10.16%	-3.68%	7.25%	25.43%	40.31%	57.76%	15.56%	16.32%	0.953
MM (Constr3):	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MinVar	12.91%	-4.22%	-2.50%	-0.52%	-0.09%	3.30%	9.10%	-1.38%	29.76%	36.72%	16.92%	7.54%	11.36%	0.664
MaxSharpe	-93.84%	8.18%	7.93%	10.73%	29.77%	10.16%	-3.68%	7.25%	25.43%	40.31%	57.76%	15.56%	16.32%	0.953
MM (Constr4):	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MinVar	1.45%	0.00%	0.00%	0.00%	0.00%	3.40%	9.36%	0.00%	30.61%	37.78%	17.40%	8.39%	11.53%	0.727
MaxSharpe	0.00%	0.65%	5.26%	1.97%	14.35%	0.00%	0.00%	0.00%	12.18%	23.71%	41.89%	12.03%	13.80%	0.872
MM (Constr5):	SPX	QCOM	AKAM	ORCL	MSFT	CVX	XOM	IMO	KO	PEP	MCD	Return	StDev	Sharpe
MinVar	0.00%	-3.35%	-2.25%	0.49%	1.60%	4.92%	10.62%	-0.53%	31.37%	38.66%	18.48%	7.88%	11.40%	0.692
MaxSharpe	0.00%	1.72%	5.79%	3.25%	16.92%	-1.37%	-13.78%	1.01%	14.39%	26.73%	45.34%	12.91%	14.59%	0.885

Despite having more significant risk exposures, MaxSharpe portfolios tend to demonstrate higher potential returns than “MinVar” portfolios. This proves the fundamental truth that higher returns typically depend on taking on greater levels of risk. Remarkably, portfolio performance exhibits outstanding consistency among different constraints in each model - attesting to the robustness of optimization approaches. Index Model portfolios designed to replicate an index within certain restrictions tend to yield lower returns and associated risks when compared with Markowitz Model portfolios, reflecting their emphasis on risk reduction for consistency rather than aggressive returns.

Conclusion

The project demonstrates the use of Modern Portfolio Theory (MPT) to optimize investment portfolios with models such as the Full Markowitz Model (MM) as well as the Single Index Model (SIM). Several fundamental observations have come to light through analyzing the construction of portfolios with various constraints. Limitations significantly influence the design and properties of minimal variation and practical frontiers, directly impacting the risk-return tradeoffs. Additionally, the decision between models has a significant impact, with the SIM constantly proving superior in these stocks’ circumstances. The two models produce nearly identical graphs to help investors find the boundary of their portfolios with minimal variance. This is typically the case for the stocks or an SPX index. That means if investors want to invest, regardless of the option, the model is incredibly adaptable and fully supports our hypothesis that the two analysis techniques are roughly identical.

For ordinary investors and not professionals, the one index model is suitable. A simplified covariance calculation can reduce the need for estimates for exponential models. The project of portfolio analysis comes with two significant disadvantages. The portfolio can be utilized to help decide on investments. However, more accurate information is needed to provide better analysis and reduce errors in operation, so the study may not be real. Second, the steps used to analyze and the formulas used seem precise pretty; however, they do not translate into a 100% solid portfolio strategy returning to the real world since many different variables influence the analysis results simultaneously.

The research findings from this project emphasize the necessity of considering real-world constraints before deciding on optimization models to build investment portfolios. These methods can improve risk management and higher returns in an ever-changing and complex financial environment. By integrating historical data with the principles of MPT, this project provides important insights to financiers and financial professionals looking to understand the complex world of portfolio optimization.

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