The Analysis of Statistical and Financial modeling based on Fourier Transform

Yang Xu

Abstract

In this manuscript, we mainly focus on the contribution of the Fourier transform in studies of economics and statistics and show how this mathematical technique helps analysts build various analytical models to understand complex data behavior. We focus on explaining how the application of the Fourier transform is used in time series analysis and regression analysis with examples and demonstrating why the Fourier transform contributes to these two analyses. First, modeling techniques without using Fourier transform omit information in time series analysis, especially when dealing with large data. Second, utilizing the Fourier Transform can sufficiently decrease the complexity of calculation and provide estimation, such as covariance estimation, in stochastic process and time series analysis.

Keywords: Fourier Transform; Autoregressive Model; Statistical analysis; Financial Models

Introduction

Admittedly, from a historical perspective, the Fourier transform has greatly contributed to physics and engineering. Still, its applications in such fields are not emphasized in this manuscript. Instead, we try to show the importance of the Fourier Transform in statistics and finance[1]. With the development of the current society, the importance of data analysis and estimation in finance and statistics is constantly magnified, especially when facing a large amount of complex data, such as bank transaction history. Bank flow data is often huge and complex, making analysis and estimation particularly complicated[2], but analysts have also discovered mathematical properties such as linear regression and frequency[3]. In such cases, many mathematical techniques become very helpful in establishing a comprehensive statistical model. Fourier transformation is one of the most powerful techniques that simplifies complexity in such situations.

This, possibly one of the most renowned mathematical techniques, was introduced by Jean-Baptiste Joseph Fourier in 1821. However, the development and perfection of the Fourier transform are not only Fourier's contribution; many well-known mathematicians, such as Leonhard Euler and Daniel Bernoulli, have also made many outstanding contributions to this technique[4]. The Fourier transform has always been a very important mathematical technique not only in mathematics studies but also frequently performed in physics and engineering. While making many scientific contributions, the Fourier transform is a relatively concise technique because its mathematical properties help researchers decompose a complex and perhaps ever-changing function into its basic sine and cosine functions[5]. This fundamental property's most representative and well-known practical applications are signal analysis and simulation[6]. Section two introduces the notation, operation, and fundamental concepts of the Fourier Transformation and Auto-regressive model. Then, in the next section, our analysis focuses on applying the Fourier Transform in financial models and proves the validity of this operation. We discuss further how these combined techniques and resulting models can be used in related fields and their efficiency.

Materials and Methods

A. Fourier Series

Fourier series (Equation 1) uses mathematical methods to express periodic functions as combinations of sine, cosine, and complex exponential functions:

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l} \tag{1}$$

Where the Fourier coefficients are given by:

$$c_n=\frac{1}{2l}\int_{-\infty}^{\infty} f(x)e^{-in\pi x/l}dx \tag{2}$$

B Fourier Transform

Fourier series only apply to periodic functions, and non-periodic functions require Fourier transforms to express them in a similar form. An integral symbol replaces the summation symbol in the Fourier series to represent non-periodic functions, and this integral is called the Fourier transform integral (Equation 3):

$$f(x)=\int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}d\alpha \tag{3}$$

Where the Fourier transform of f(x) is given by:

$$g(\alpha)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}dx \tag{4}$$

The functions f(x) and g(α) are Fourier transforms of each other.

The Fourier transform can be compared and contrasted with the Fourier series. In the Fourier series, the sine and cosine functions cover a periodic function's odd and even parts. Similarly, in non-periodic functions, the Fourier transform is divided into sine and cosine transforms. The Fourier sine function (Equation 5) covers the odd part of a non-periodic function:

$$f_{\text{sine}}(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g_{\text{sine}}(\alpha)\sin\alpha x d\alpha \tag{5}$$

Where the Fourier sine transform of f(x) is given by:

$$g_{\text{sine}}(\alpha)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f_{\text{sine}}(x)\sin\alpha x dx \tag{6}$$

Similarly, the Fourier cosine function (Equation7) covers the even part of a non-periodic function:

$$f_{\text{cosine}}(x)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} g_{\text{cosine}}(\alpha)\cos\alpha x d\alpha \tag{7}$$

$$g_{\text{cosine}}(\alpha)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty} f_{\text{cosine}}(x)\cos\alpha x dx \tag{8}$$

Where the Fourier cosine transform of f(x) is given by:

The Fourier Sine Transform and Fourier Cosine Transform exist in pairs: transforms and inverse transforms.

C Discrete Fourier Transform (DFT) and Fast Fourier Transform (FFT)

In addition to simple Fourier Transforms, the application of Discrete Fourier Transformation (DFT) (Equation 9) is commonly used in statistical modeling and financial analysis[7]. Compared to Fourier Transforms, the DFT maps a sequence of data (points) to complex functions of frequency, which are functions of period (time). Since DFT deals with discrete inputs, it is not an integral but a summation of sets of complex numbers:

$$X_k=\sum_{n=0}^{N-1} x_n e^{-2\pi i/Nkn} \tag{9}$$

Xk is the complex amplitude at frequency k, xn is the input data point at time n, N is the number of input data points, and e−2πi/Nkn is the complex exponential function. This equation represents the Discrete Fourier Transform.

Like the Fourier Transform, the Discrete Fourier Transform also has an inverse function. The Inverse Discrete Fourier Transform is:

$$x_n=\frac{1}{N}\sum_{k=0}^{N-1} X_k e^{i2\pi nk/N} \tag{10}$$

Where xn is the nth sample of the time-domain Signal, Xk is the kth sample of the frequency-domain Signal, N is the total number of samples, and i is the imaginary unit.

In general, Fourier Transform deals with continuous functions and decomposes them into a combination of sine and cosine functions. In contrast, Discrete Fourier Transform processes finite sequences of inputs. Discrete Fourier Transform can transform finite and equally-spaced data into complex-valued functions. Discrete Fourier Transform is important in this manuscript because, in statistical and financial analysis, most of the data that analysts encounter are individual data points, such as bank transaction records, and rarely are they a single data point with a clear expression. The reason for the importance of the Discrete Fourier Transform in this article is that most of the data analyzed in the field of statistics and finance analyze individual data points, such as bank transaction records, instead of continuous functions.

In addition, in statistical analysis, the input data to be computed and analyzed can be very large, making the computation of the Discrete Fourier Transform complex and time-consuming. However, since Discrete Fourier Transform is a method dealing with finite data, algorithms can do it. Such algorithms used to compute the Discrete Fourier Transform and Inverse Discrete Fourier Transform are called Fast Fourier Transforms[8].

D Auto-regressive (AR) Model

The autoregression model is based on the evolution of the linear regression model in regression analysis. The usual application of AR models in statistics is a method for processing and analyzing time series[9]. The AR model outputs predicted values that are linearly related to the input values by combining past (occurred) input data with a stochastic process. Because the AR model considers stochasticity, its output values are not uniform like the solutions of differential
equations.

An auto-regressive model of order ρ is noted as:

$$y_n=c+\sum_{i=1}^{p}\phi_i y_{n-i}+\epsilon_n \tag{11}$$

In polynomial notation:

$$y_n=c+\phi_1 y_{n-1}+\cdots+\phi_p y_{n-p}+\epsilon_n \tag{12}$$

Where yn is the nth observation of the time series, c is a constant, p is the order of the AR model, φi are the coefficients of the lagged observations, and εn is white noise with zero mean and variance σ2. The AR model of order p is denoted as AR(p)[10]. A typical autoregressive model with p = 1 (FIG. 1) is here.

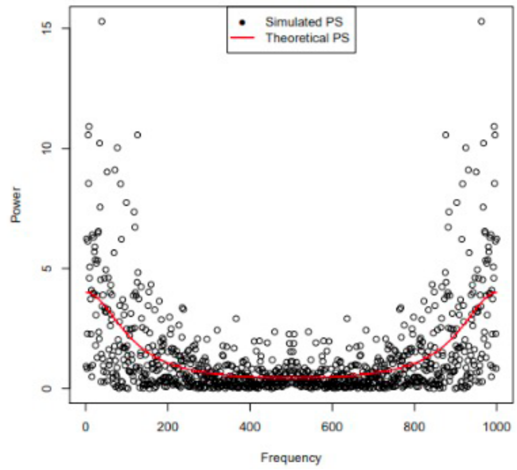

Fig.1 Spectrum of AR(1) Process

Analysis

In the paper "Examining Applications of Fourier Transforms to Financial Data and Covariance Estimation," the author performs an experiment that compares the estimation calculated through two models: the traditional autoregression model and the modified with Fourier Transformation.

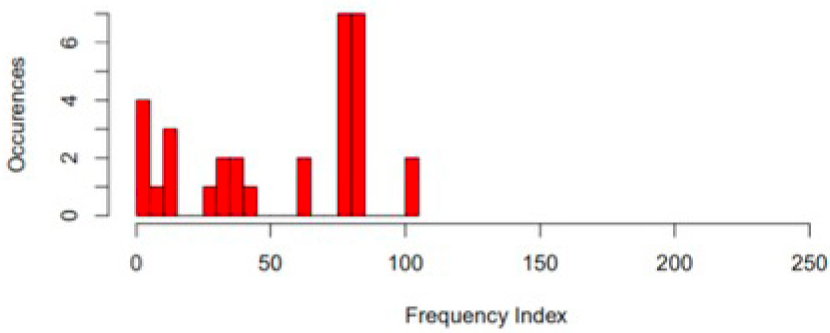

Fig.2(a): AR(1) Residual Frequencies Significant (5 %)

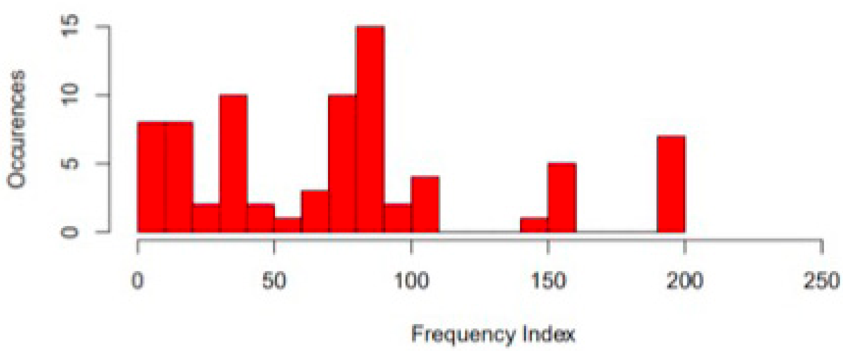

Fig.2(b): AR(1) Residual Frequencies Significant (10%)

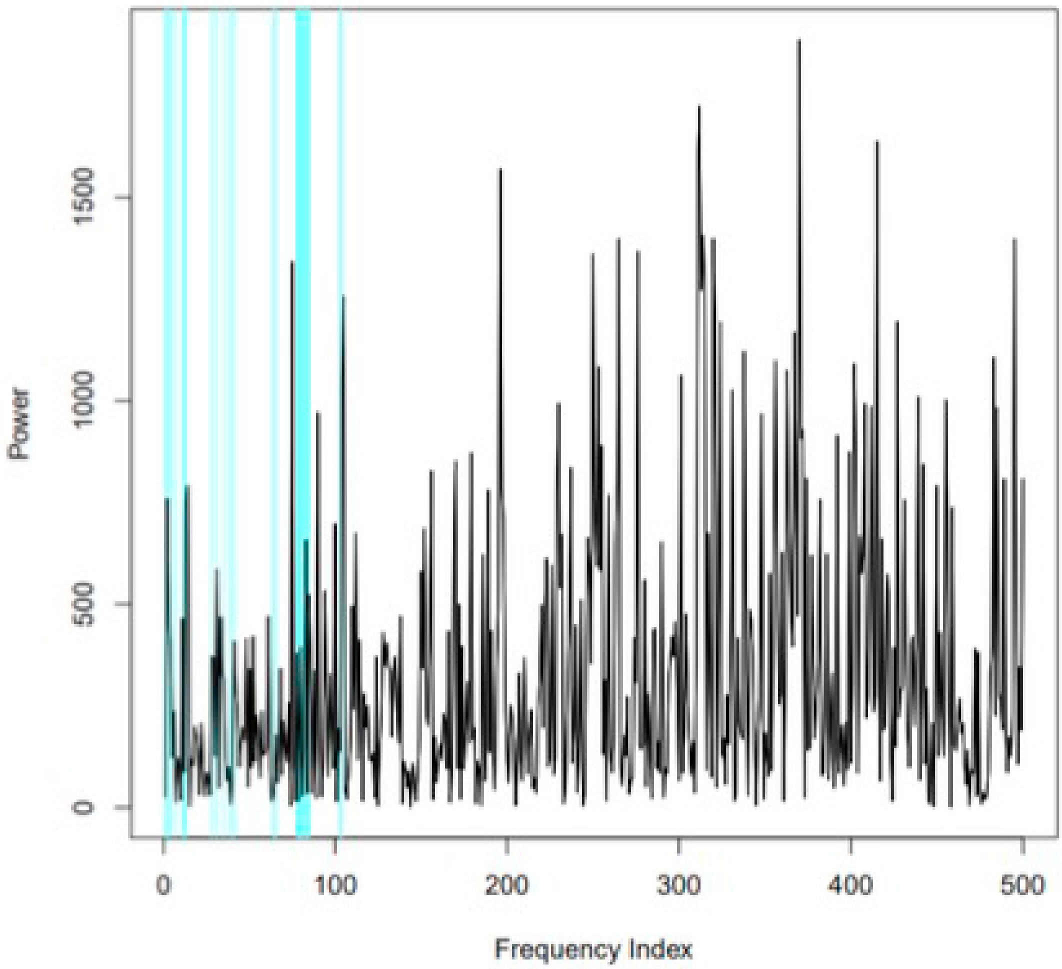

Fig.2(c): Blue Bands Indicate Frequencies Determined Significant (5%) with Regression Analysis

Fig.2(c): Blue Bands Indicate Frequencies Determined Significant (5%) with Regression Analysis After analyzing the outputs, the experimentalist concludes that the model with Fourier Transforms captures periodicity that is not observed in the autoregression model (Fig. 2.a and Fig. 2.b). Significant frequencies were observed in the Autoregression model compared to the analysis observed in Power Spectrum (Fig. 2.c).

Furthermore, by analyzing the pattern of outcomes of the Autoregression model, the experimentalist also noticed that the significance of frequencies exists in the calculation. By these frequencies, the estimation required for data points is sufficiently reduced. The paper also justifies the usability of Fourier Transformation in regression analysis by providing evidence of periodicity and cyclicality that occurred in the Fourier Series of time series analysis[11].

The result indicates that using Fourier Transform in modeling is valid, and its application indeed enhances data estimation. In the paper "Frequencies in Ultra-high-frequent Trading," the experimentalist compares traditional time series analysis and Fourier transformed power spectrum similarly. The outcome of such a comparison also finds the significance of frequencies, and such frequency contributes to a clearer estimation compared to traditional methods (Fig. 3.a and Fig. 3.b). In addition, finding Fourier coefficients under the circumstances of decomposing time series simplifies the complicated calculation in high-frequency trading estimation [12].

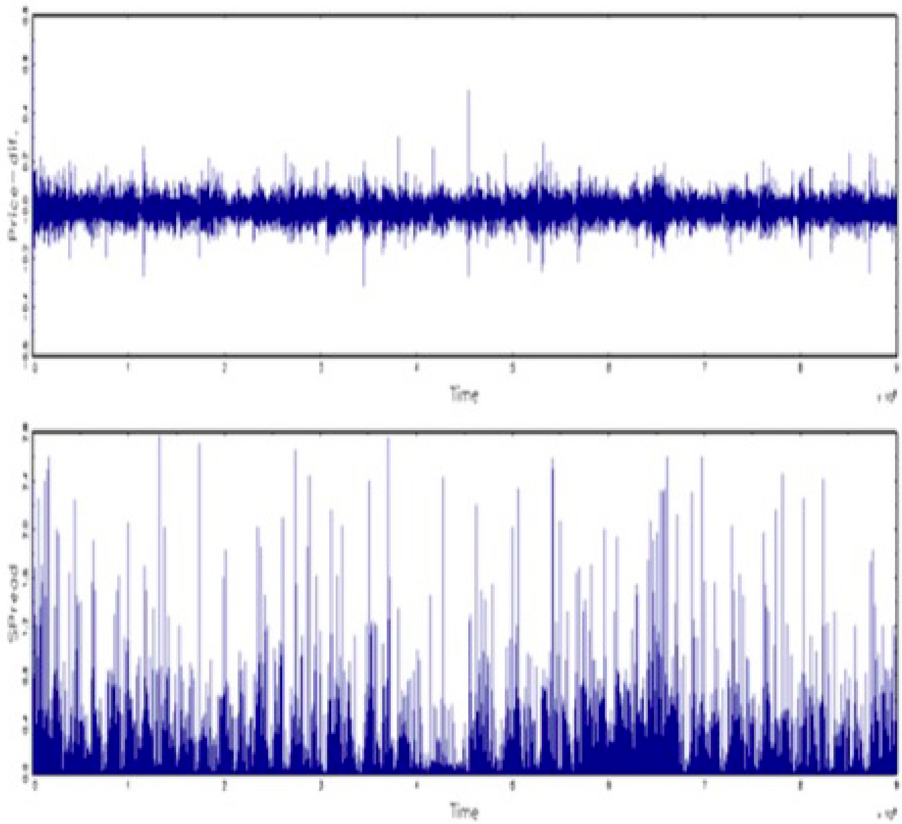

Fig.3(a): The original （irregularly spaced） time series

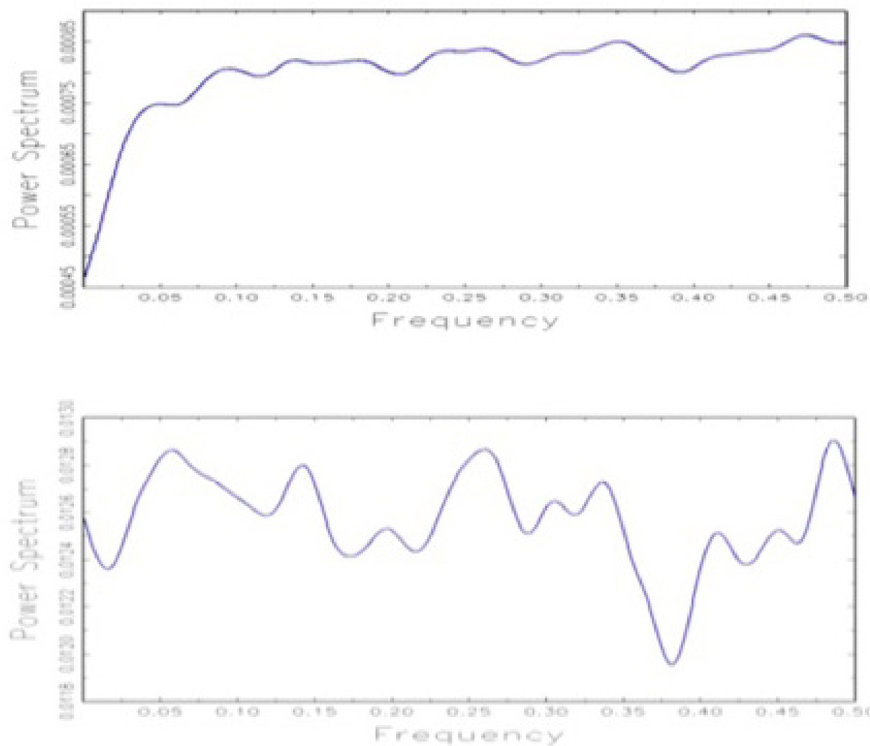

Fig.3(b): Comparison between traditional time series analysis and corresponding operated power spectrum.

In the residual analysis of frequencies, we show the validity that performing the Fourier transform can capture periodic variation, which is not emphasized by traditional autoregression data. In residuals analysis, using the traditional autoregression model, we observe the significance of frequencies in the diagram. Further applying power spectrum analysis, we observed that the frequencies we obtained here highly coincide with those we previously observed. We can conclude that there is a specific variation occurring in the residuals that the existence of frequencies can explain. Therefore, applying Fourier transformation in financial data is valid and can provide analyst information that cannot be shown through the traditional regression model.

In addition, Fourier transformation is highly useful when the data is large since it helps reduce computational complexity. By analyzing the performance of the autoregression model and Fourier analysis methods in high-frequency trading, we can observe that the latter techniques are more efficient and can comprehensively investigate the data.

Discussion

This manuscript demonstrates that applying the Fourier transform in statistics and finance is reasonable through its convenience and information completeness in the context of time series. Frequencies inevitably exist in time series analysis and regression analysis. On this basis, we can obtain more useful information by applying the Fourier transform to these frequencies. In addition, the results imply that the application of Fourier transformation in statistical analysis is valid and contributes to a more comprehensive and efficient observation compared to the traditional regression analysis model.

The application of the Fourier transform in statistics is presented in linear regression analysis and has significant effects in spectral analysis, power spectrum estimation, and dealing with nonparametric statistics. For example, in the analysis of earth environment data and some psychological data, the frequency-related information obtained by the Fourier transform can be used to reflect the evolution of the situation and to predict and analyze possible future scenarios for analysts. In addition, when dealing with data that lack significant distribution, the Fourier transform of different parameters sometimes allows analysts to capture more relevant information to understand the data further and make corresponding decisions.

The Fourier transform is a representative method not only in statistics and finance but also in physics, computer science, medicine, and many other science fields, where it is very important to human society. One of the most famous applications is in signal processing: Fourier transform manipulates communication, radio, and even biomedical signals to eliminate unwanted signal bands and further analyze the related activities. For example, the MRI or CT brain scan[13]. In addition, the Fourier transform has important and relevant contributions in computer science, such as image processing and cryptography, and physics, such as the analysis of waves and vibrations, which contribute substantially to today's society.

Although Fourier's transformations have made significant contributions in many fields, they still have limitations in some areas. First, the resolution of the Fourier-transformed results is limited. This resolution is related to the length of the input data and the sample density, which will lead to limitations in the Fourier transform output when analyzing data at high time frequencies. Second, the assumptions obtained from the Fourier transform are stationary, which means that using the Fourier transform alone may not yield the desired outcome predictions in some data and processes with significant stochastic processes. In addition, the Fourier transform provides a frequency domain that is sometimes difficult to analyze. If the right signal processing analysis method is not chosen, then the predictions may be inaccurate.

With the development of computer technology, the algorithms related to the Fourier transform especially the fast Fourier transform, have become more sophisticated. This may increase analysts' feasibility of using Fourier transforms in more scenarios, especially those models that include complex computational processes. This may further allow analysts to capture more information.

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