Numerical Methods for Option Pricing

Bohan Jiang

Abstract:

Nowadays, financial markets have become more complex and have given rise to more research opportunities, one of which focuses on research related to the pricing of various financial instruments used in options trading. In this essay, a gradual process of reflection is used to deepen the option pricing theory. Firstly, an effective mathematical method for pricing option contracts is using the binomial tree model, a relatively straightforward way of calculating the value of an option. However, it has only two upward and downward trends and has limitations that are the most different from the actual market conditions. That is why the study will have to continue in-depth to get as close as possible to the real situation in the market. The next step is to make the model more in line with market reality by adding a possible rate of change, resulting in the Trinomial Tree Model. The model incorporates the possibility of future price increases, decreases, or stabilization (the third change). The Trinomial Tree Model follows the no-arbitrage principle and removes the assumption of a risk-free market opportunity. This paper derives a method for constraining market prices under risk-neutral conditions. This is crucial for investors seeking profitable outcomes in options trading. The main objective of this research-based paper is to develop a complete theory of option pricing in a one-step trinomial tree model.

Keywords: Trinomial tree model, completing market, pricing option.

1. Introduction

Option pricing methods are discussed in the study of complex financial market changes and the appreciation and depreciation of stocks and bonds under different market scenarios under the basic condition of no arbitrage. In work, group discussions and mathematical operations are carried out for different ranges of values in the completing market, and conclusions are drawn at the end.

2. Binomial Tree Model

The binomial tree model is an effective method to determine option prices. The model was first introduced by Cox et al., Rendleman, and Bartter in 1979 [1]. In the binomial model, only two possible outcomes correspond to stock value appreciation and depreciation, respectively [2].

The following conditions are followed throughout the article. Only two primary assets are being traded in the entire market. One is the bond, the risk-free asset, and every period, it will have a certain rate of increase. The start moment of the value of the bond simplifies as B0. The certain rate of the increase simplifies as *r*, so each year, the annually changing rate is (1+r). The value for a certain period of the bond B*t*=(1+r) ^t *B0. (t is the number of the step of the binomial tree model). Another primary asset is stock. The value of the stock can be divided into two situations since the value of the stock can rise or go down. The probability of the stock appreciation is denoted by *p*, then the probability of the depreciation is 1−p. The value of the stock is *Sj*, and the next step has p probability be the *S*(*j*−1)*u*, and 1-p probability be the *S*(*j*−1)*d* [3].

Assume at time one (t = 1), the total value of the asset:

$$X_1 = N^B B_1 + N^S S_1 = \begin{cases} N^B(1+r) + N^S S_0 u, & \text{with probability } p \\ N^B(1+r) + N^S S_0 d, & \text{with probability } 1-p \end{cases} \tag{1}$$

The wealth *X*1 is equal to the option's payoff *S*(*S*1). Then, we have the following two equations:

$$\begin{cases} N^B(1+r) + N^S S_0 u = F\left(S_0 u\right) \\ N^B(1+r) + N^S S_0 d = F\left(S_0 d\right). \end{cases} \tag{2}$$

Combining these two equations can obtain:

$$\begin{cases} N^B = \dfrac{1}{1+r}\dfrac{uF\left(S_0 d\right) - dF\left(S_0 u\right)}{u-d}, \\ N^S = \dfrac{F\left(S_0 u\right) - F\left(S_0 d\right)}{S_0(u-d)}. \end{cases} \tag{3}$$

We set *q*=1+r−d/u−d. Then we express the investment *X*0 required to set up this portfolio at time 0 as:

$$X_0 = \frac{1}{1+r}\left(qF\left(S_0u\right)+(1-q)F\left(S_0d\right)\right). \tag{4}$$

However, the binomial tree model is far from the actual situation, is less efficient, and does not help the user to get pricing effectively, so the next part will move to the trinomial tree model. In the next model, there is one more fork, which leads to faster convergence, thus helping the user get the desired result faster. We will move to the second main part, the trinomial tree model.

3. Trinomial tree model

The S (0) and S (1) are defined to be the stock price at time 0 and 1, at present, the time t=0. Assuming there are three cases at time t, which is stock price going up to S (t) =S (0) u, stock price goes to S (t) = S_0m , or stock price goes down to $S(t)$= S_0d , respectively. And F (S_0u) is defined be the option price when stock price is , the aiming of this part is to price the call option on time 0 defined as X_0 , r is risk free interest rate [4].

Then, the value of S (1) can be expressed as follows:

$$S(1)=\begin{cases} S(0)u, \text{ with probability } P_u \\ S(0)m, \text{ with probability } P_m \\ S(0)d, \text{ with probability } P_d \end{cases} \tag{5}$$

Probability together is 1.

The payoff obtained by purchasing NB units of bond with NS units of stock replicates the same payoff as purchasing the option. When the stock price rises by a u amount of change, the second equation when there is a m amount of change in the value of the stock (uncertainty of the price goes up, down, or no change), and the third equation when the value of the stock falls by a d amount. And the simulation of this portfolio holds when the starting price is equal to the price of the option's payoff at this stage. By using the replication strategy (NB, NS), we obtain three equations:

$$\begin{aligned} N^B(1+r)+N^S S_0u &= F\left(S_0u\right), \\ N^B(1+r)+N^S S_0m &= F\left(S_0m\right), \\ N^B(1+r)+N^S S_0d &= F\left(S_0d\right). \end{aligned} \tag{6}$$

Unfortunately, equations with two unknowns and three equations do not always have a. solution. Indeed, there are two variables NB, NS with three equations; therefore, it is not always possible to solve these equations and determine the option price this way.

It is calculated that there can be a solution only if the option price is actually within this range:

$$\frac{1+r-m}{u-m}S_0(u-m) \leqslant X_0 \leqslant \frac{1+r-d}{u-d}S_0(u-m) \tag{7}$$

4. Completing the market

Previously, in the last part, the research focused on an incomplete market characterized by two variables (NB and NS) and three equations. A further primary asset is introduced to complete the market, which in turn has three equations and three unknowns, making the unsolved equation solvable. In a complete market, items can be traded continuously without intermediate loss [5].

The amount of change in the price of the new option under the newly introduced asset has to be classified and discussed. The amount of change in the new asset, w, is classified into four cases: w is greater than u, u is greater than w is greater than m, m is greater than w is greater than d, and w is less than d. However, given the limited level of computation and mathematical skills, the results are still subject to further consideration.

Acknowledgement:

I would like to thank Johannes Ruf, Professor in the Department of Mathematics at LSE, for his guidance on the direction of the thesis and for revising the details.

Thanks to Zijun Zhang for his contribution to the mathematical operations in the group project and Hang Su for his contribution to the narrative descriptions.

References

[1] Korn, R. and Muller, S. (2010a). *Recent Developments in Applied Probability. and Statistics: Dedicated to the Memory of Jürgen Lehn*. [online] scholar.google.com. Available at: https://scholar.google.com/scholar?hl=zh-CN&as_sdt=0%2C5&q=history+of+binomial+tree+model&btnG=#d=gs_qabs&t=1696673413811&u=%23p%3Da-Ss_Y5zuIgJ [Accessed 7 Oct. 2023].

[2] Paul Clifford, Yan Wang, Oleg Zaboronski. Pricing options using trinomial trees. Working paper, available from https://warwick.ac.uk/fac/sci/maths/people/staff/oleg_zaboronski/fm/trinomial_tree_2010_kevin.pdf (retrieved on August 1st, 2023). 2010.

[3] Fei Lung Yuen, Hailiang Yang. Option pricing with regime switching by. trinomial tree method. *Journal of Computational and Applied Mathematics*. Volume 233, Issue 8, Pages 1821-1833. 2010.

[4] John Hull. Options, Futures, and Other Derivatives (Tenth edition). Harlow, UK Pearson. 2018.

[5] Platen, E. (2002). Arbitrage in continuous complete markets. Advances in Applied Probability, 34(3), pp.540–558. doi:https://doi.org/10.1239/aap/1033662165.