Empirical analysis of optimized portfolio allocation based on Markowitz and index models

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Abstract:
There is a need to achieve a balance between asset returns and risks. This has remained a central focus of financial market research. It has served as a crucial reference for investment decision-making. There has been a weakness as this financial investment theory relies on qualitative analysis. This method has lacked robust quantitative methods. The resurgence and expansion of Western economies have led to a flourishing financial investment activities. This has prompted the emergence of the Modern Portfolio Theory (MPT). The theory was pioneered by Harry Markowitz in 1952. MPT has rapidly evolved over the period. It has attracted numerous scholars and yielded substantial research outcomes (Markowitz, 1991).
Keywords: Markowitz, quantitative, investment, risk

1. Introduction

There is a need to achieve a balance between asset returns and risks. This has remained a central focus of financial market research. It has served as a crucial reference for investment decision-making. There has been a weakness as this financial investment theory relies on qualitative analysis. This method has lacked robust quantitative methods. The resurgence and expansion of Western economies have led to a flourishing financial investment activities. This has prompted the emergence of the Modern Portfolio Theory (MPT). The theory was pioneered by Harry Markowitz in 1952. MPT has rapidly evolved over the period. It has attracted numerous scholars and yielded substantial research outcomes (Markowitz, 1991).

The researcher led to a groundbreaking contribution, establishing the foundation for Modern Portfolio Theory. This led to the introduction of the concept of mean-variance portfolio selection. The model seeks to optimize portfolio selection by considering the mean and variance of asset returns. Investors aim to either maximize the expected rate of return for a given level of risk or minimize the variance for a given expected return. Markowitz's approach derivates the portfolio efficient frontier, allowing investors to tailor their portfolios according to their risk preferences. This theory underscores the importance of considering individual asset characteristics and their statistical interactions to diversify individual risks and retain primarily systematic risks (Fabozzi et al., 2008).

However, the complexity of the full mean-variance model proposed by Markowitz, demanding numerous estimates, has led scholars to focus on the simplified Index Model introduced by William Sharpe around a decade later. The Index Model, presented in 1963, streamlines the solution process of the mean-variance model, facilitating its practical application to portfolios of varying sizes. Index models, often referred to as factor models, find widespread application in stock portfolio selection and allocation, while standard mean-variance models are preferred for smaller portfolios.

Amidst the continuous expansion of financial markets and the growing significance of stock markets in national economies, the issue of significant stock price volatility has emerged. This raises questions about the efficacy of classic portfolio selection models in the current market environment. To address this concern, this paper systematically examines the application of the Markowitz model and the index model in the current market. It explores both models' practical investment performance and application methods, simulates real investments to obtain optimal portfolios, imposes relevant constraints for different market conditions, and analyzes the variations in portfolio construction under diverse environments. This research aims to deepen our understanding of the classical Markowitz model, guide investors toward more informed investment decisions, foster the healthy growth of the stock market, maintain financial stability, and enhance the role of finance in serving the real economy (Zanjirdar, 2020).

1.1 Aim of the Research

The paper's main objective was to apply the principles of modern portfolio theory, specifically Markowitz's optimal portfolio selection and the simplified index model. Utilizing 20 years of historical daily total return data from

Yahoo Finance, we focused on ten stocks categorized into four distinct sector groups, an (S&P 500) stock index, and a surrogate for the risk-free interest rate (1-month federal funds rate). Employing monthly data, we computed relevant optimization inputs for both the full Markowitz model (MM) and the index model (IM). We identified additional constraints governing portfolio regions with these optimization inputs, including the efficient boundary, minimum risk portfolio, optimal portfolio, and minimum return portfolio boundary for the five constraints. Subsequently, we conducted a comprehensive analysis to compare the diverse constraints for each optimization problem (MM and IM) and for each optimization problem between the two solutions, considering identical constraints.

2. Method

2.1 Data and Variables

Daily data of the total returns of 10 stocks and the S&P 500 index from 5/11/2001 to 5/12/2021 was obtained from Yahoo! Finance. Table 1 below shows the stocks and the sectors in which the companies operate.

Table 1: Data Variables

Ticker Symbols	Full Name of Company	Sector
NVDA	NVIDIA Corporation	Technology
CSCO	Cisco Systems, Inc	Technology
INTC	Intel Corporation	Technology
GS	The Goldman Sachs Group, Inc	Financial Services
USB	U.S Bancorp	Financial Services
TD CN	The Toronto-Dominion Bank	Financial Services
ALL	The Allstate Corporation	Financial Services
PG	The Procter & Gamble Company	Consumer Defensive
JNJ	Johnson & Johnson	Healthcare
CL	Colgate-Palmolive Company	Consumer Defensive

In addition to the above stock variables, we used S&P 500 index (SPX) and Risk-free rate (FEDL01). The market index benchmarks the overall stock price movements and performance.

2.2 Models

2.2.1 Markowitz Model (MM)

The Markowitz Model is a mathematical approach to building investment portfolios to optimize the balance between risk and return. The core principle involves diversification, where combining assets with varying risk and return profiles results in a more efficient portfolio. The model incorporates key elements such as expected return, measured as a weighted average of individual asset returns; risk, quantified by portfolio variance considering both individual asset variances and correlations; the efficient frontier, representing portfolios offering the best risk-return trade-offs; and the integration of a risk-free rate, facilitating the construction of the Capital Market Line (CML) and the optimal risky portfolio. The Capital Allocation Line (CAL) demonstrates the trade-off between risk and return for a mix of a risk-free asset and a risky portfolio, with the slope indicating the Sharpe ratio. Despite criticisms for assumptions and sensitivity to input parameters, the Markowitz Model has significantly influenced the finance and portfolio management domains (Guerard Jr, 2009).

Mathematical Presentation of MM

MM Portfolio return is given as;

Where refers to the unknown set of instruments' weights and is the set of instruments' average returns.

MM Portfolio standard deviation is given by;

Where is an auxiliary vector given by $\{\}^T$ and P is the matrix of instruments' cross-correlation coefficients.

2.2.2 Index Model (IM)

The Index Model can be described as the financial theory that builds upon the ideas of the Markowitz Model. William Sharpe developed the model in the 1960s. It provides a simplified version of estimating an individual asset's expected return and risk within a portfolio. The model has a key assumption that makes it unique. The model's returns of individual assets are influenced by both systematic risk (market risk) and unsystematic risk (specific to the asset). The model emphasizes systematic risk, which captures the overall market's performance. This is typically represented by a market index like the S&P 500. In this framework's context, an asset's expected return is a function of its sensitivity to market movements. This is measured by the asset's beta and the market's expected return. The beta coefficient reflects how much an asset's returns are expected to move in response to changes in the market. The model has simplified the diversification process. This is due to the market risk component and how it helps investors in the decision-making process based on the risk-return profile of an asset about the broader market (Jin et al., 2021).

Mathematical Presentation of IM

IM Portfolio return;

IM portfolio standard deviation;

Where is the portfolio beta?

2.3 Constraints

The Excel solver tool can optimize IM and MM portfolio problems under the following constraints.

1.

2.

3. No constraints

4.

5.

3. Results

Table 2: Portfolio Returns and Sharpe Ratio Under MM and IM

	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MM	-38.5%	23.3%	-1.8%	-9.7%	1.0%	5.1%	36.4%	3.8%	53.0%	26.1%	1.4%	16.0%	16.3%	0.980
IM	0.0%	8.3%	-6.0%	-4.6%	-6.8%	2.5%	23.3%	0.4%	39.0%	26.4%	17.5%	11.0%	11.7%	0.943

Table 3: Minimal Variance and Maximal Sharpe under MM

MM (Constr1):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	38.37%	-2.97%	-2.89%	1.33%	-5.90%	-0.30%	19.41%	-11.48%	25.93%	18.83%	19.67%	7.51%	10.95%	0.685
MaxSharpe	-42.74%	15.75%	-1.15%	-6.11%	3.25%	6.48%	35.29%	1.07%	45.71%	30.00%	12.45%	14.01%	13.95%	1.004
MM (Constr2):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	38.37%	-2.97%	-2.89%	1.33%	-5.90%	-0.30%	19.41%	-11.48%	25.93%	18.83%	19.67%	7.51%	10.95%	0.685
MaxSharpe	-100.00%	21.50%	0.31%	-8.15%	11.46%	12.25%	44.92%	6.87%	52.33%	41.02%	17.48%	16.56%	16.06%	1.031
MM (Constr3):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	38.37%	-2.97%	-2.89%	1.33%	-5.90%	-0.30%	19.41%	-11.48%	25.93%	18.83%	19.67%	7.51%	10.95%	0.685
MaxSharpe	-109.97%	22.46%	0.89%	-8.19%	12.73%	13.21%	46.46%	7.90%	53.50%	42.72%	18.30%	16.99%	16.48%	1.031
MM (Constr4):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	9.49%	0.00%	0.00%	0.00%	0.00%	0.00%	19.85%	0.00%	28.91%	20.62%	21.13%	8.88%	11.27%	0.788
MaxSharpe	0.00%	10.95%	0.00%	0.00%	0.00%	0.00%	23.73%	0.00%	42.56%	16.17%	6.60%	12.06%	13.12%	0.919
MM (Constr5):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	0.00%	-0.97%	0.08%	2.51%	-0.99%	3.50%	24.70%	-8.17%	28.91%	25.58%	24.85%	8.71%	11.18%	0.779
MaxSharpe	0.00%	14.93%	-6.85%	-10.14%	-1.30%	2.43%	30.65%	-2.27%	43.31%	23.61%	5.64%	13.06%	13.69%	0.954

Table 4: Minimal Variance and Maximal Sharpe under IM

IM (Constr1):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	25.62%	-4.04%	-5.27%	-2.81%	-8.73%	0.76%	10.28%	-1.40%	31.27%	27.68%	26.64%	7.15%	9.63%	0.742
MaxSharpe	-47.62%	8.88%	-1.24%	-0.49%	-0.64%	6.67%	29.55%	4.57%	43.95%	33.35%	23.01%	12.07%	12.18%	0.990
IM (Constr2):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	25.62%	-4.04%	-5.27%	-2.81%	-8.73%	0.76%	10.28%	-1.40%	31.27%	27.67%	26.64%	7.15%	9.63%	0.742
MaxSharpe	-70.16%	10.32%	-0.57%	-0.11%	0.57%	9.39%	34.25%	7.35%	46.90%	36.94%	25.11%	12.87%	12.92%	0.996
IM (Constr3):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	25.62%	-4.04%	-5.27%	-2.81%	-8.73%	0.76%	10.28%	-1.40%	31.27%	27.67%	26.64%	7.15%	9.63%	0.742
MaxSharpe	-70.16%	10.32%	-0.57%	-0.11%	0.57%	9.39%	34.25%	7.35%	46.90%	36.94%	25.11%	12.87%	12.92%	0.996
IM (Constr4):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	9.21%	0.00%	33.55%	28.89%	28.35%	8.64%	10.16%	0.850
MaxSharpe	0.00%	6.74%	0.00%	0.00%	0.00%	0.00%	17.75%	0.00%	37.34%	22.75%	15.41%	10.71%	11.72%	0.914
IM (Constr5):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe
MinVar	0.00%	-3.35%	-3.28%	-1.14%	-6.02%	3.27%	14.25%	1.13%	34.13%	31.54%	29.47%	7.82%	9.75%	0.802
MaxSharpe	0.00%	7.68%	-5.84%	-4.45%	-6.81%	2.56%	22.82%	0.47%	38.76%	26.69%	18.11%	10.84%	11.48%	0.944

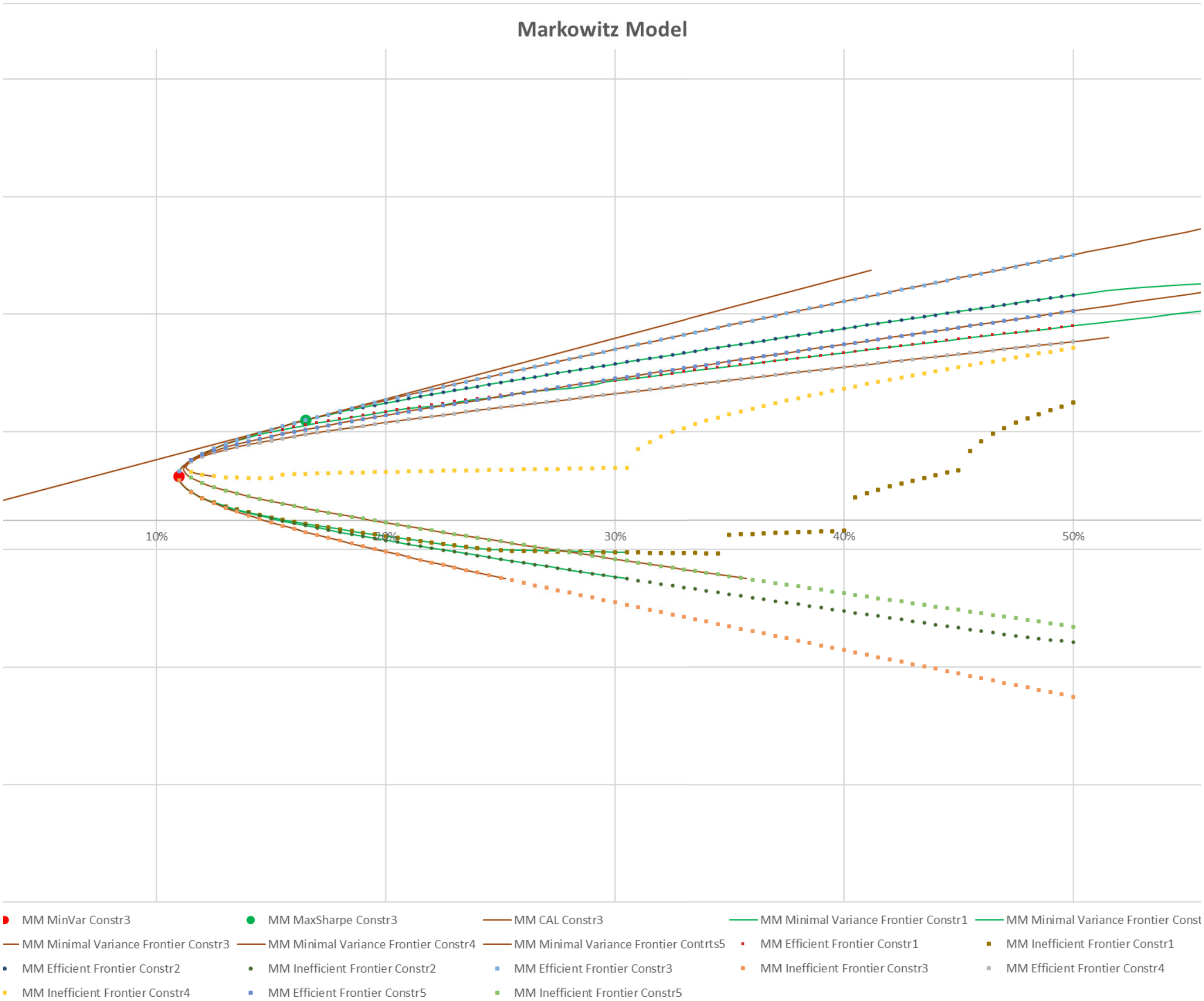

Figure 1: Efficient Frontier under MM

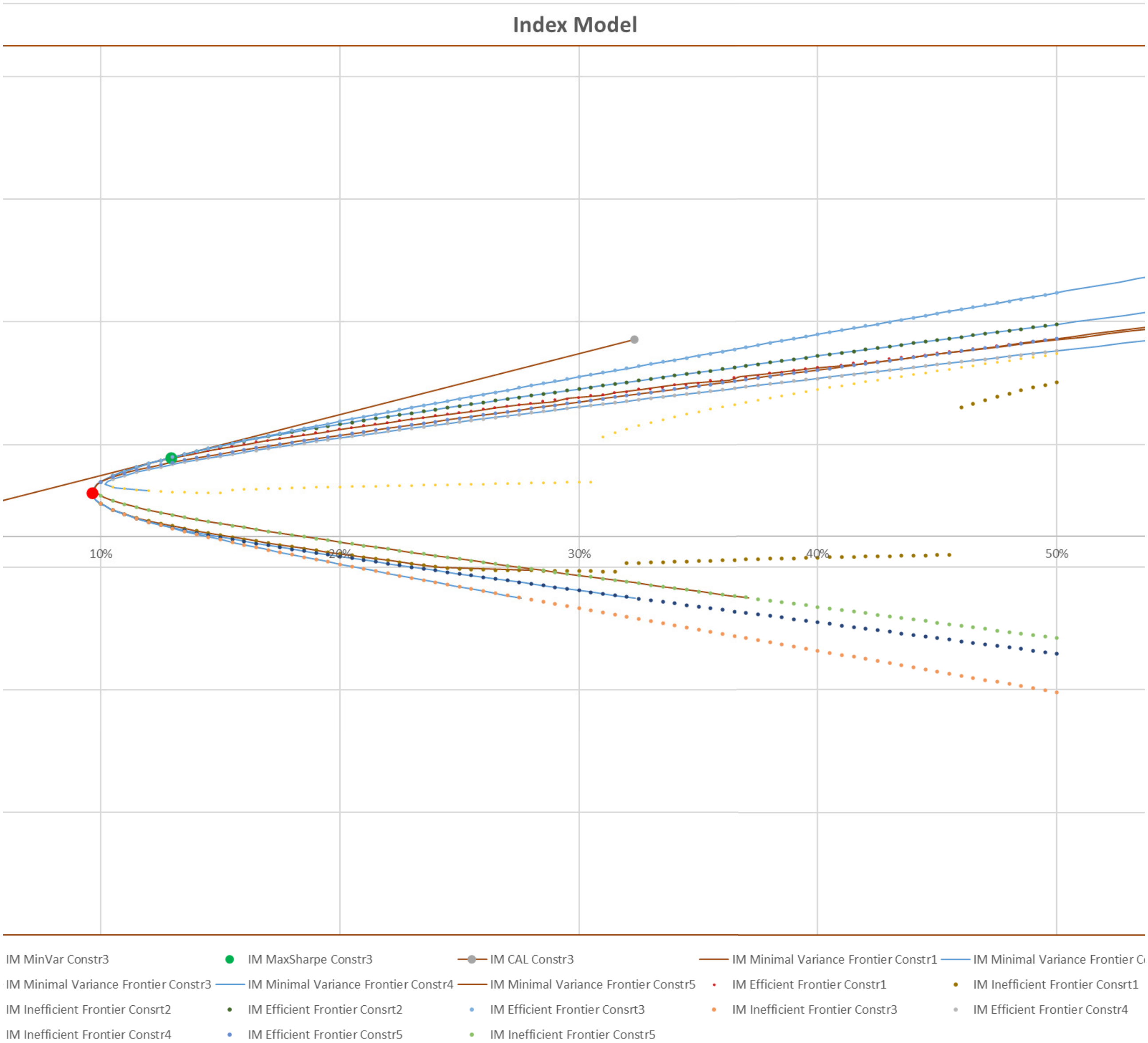

Figure 2: Efficient Frontier under IM

4. Discussion and Comparison of Models

4.1 Comparison between MM and IM models

Based on Table 2, the Markowitz Model (MM) and the Index Model (IM) have yielded the results to compare the return, standard deviation, and Sharpe ratio. The MM portfolio demonstrated a wider range of individual asset returns. These spanned from -38.5% to 53.0%, with an overall portfolio return of 16.0%. The IM portfolio exhibited less variability in returns. This ranged from -6.8% to 39.0%, resulting in an overall portfolio return of 11.0%.

Regarding standard deviation, the MM portfolio shows a broader range of 1.4% to 36.4%, with an overall portfolio standard deviation of 16.3%, and the IM portfolio's standard deviations range from 0.4% to 39.0%, with an overall portfolio standard deviation of 11.7%. While the MM portfolio has higher returns and standard deviations, the risk-adjusted performance, as measured by the Sharpe ratio, favors the MM portfolio with a ratio of 0.980 compared to the IM portfolio's 0.943. Based on these, the MM portfolio presents a superior trade-off between risk and return.

4.2 Minimal Variance and Maximal Sharpe under MM and IM

The results under the Markowitz Model (MM) and Index Model (IM) with different constraints in Table 3 and Table 4 show the trade-offs between risk and return given specific portfolio constraints. Based on the results, the following were observed: First, the minimum Variance Portfolio (MinVar) consistently has a lower return, standard deviation, and Sharpe ratio than portfolios optimized for maximum Sharpe ratio. Notably, the minimum variance portfolio is designed to minimize risk, and as a result, it sacrifices potential returns. Secondly, portfolios optimized for the maximum Sharpe ratio tend to have higher returns, higher standard deviations, and higher Sharpe ratios than the Minimum Variance Portfolio. It is also important to note that the Maximum Sharpe Ratio Portfolio aims to achieve the best risk-adjusted return, striking a balance between risk and return. Third, the comparison across constraints shows that as constraints vary, the trade-off between risk and return also changes. For example, Constr4 and Constr5 have higher Minimum Variance Portfolio returns than Constr1 and Constr3. This implies that the Maximum Sharpe Ratio of Portfolio returns generally increases across Constr1 to Constr5, indicating that relaxing certain constraints can lead to higher potential returns. The Sharpe ratio has provided a good measure of risk-adjusted performance. Based on the results, the same set of constraints, the Sharpe ratios under the MM model tend to be higher than those under the IM model. This implies that the MM portfolios offer better risk-adjusted returns within the given constraints relative to the IM portfolios.

4.3 Efficient Frontier Points for MM and IM

According to Figures 1 and 2, there is the observation that the MM portfolio points to the Efficient Frontier. It has a higher return of 17% relative to the IM portfolio. This had a return of 12.1% for the given risk level. The MM portfolio exhibited a higher level of risk, recorded at 16.5% relative to the IM portfolio, with 13% for the given level of return. This implied that investors seeking higher returns may find the MM portfolio more attractive. However, this comes at the cost of higher risk relative to the IM portfolio. This is not the same as MM, as it offers a lower level of risk for the given return. This makes it potentially more appealing to investors with a lower risk tolerance.

5. Conclusion

The comparison between the Markowitz Model (MM) and the Index Model (IM) reveals distinct characteristics and trade-offs between risk and return. The MM emphasizes diversification and optimization in the mean-variance portfolio selection. The IM simplifies the diversification process by considering the market risk component. The analysis of portfolio returns, standard deviations, and Sharpe ratios under different constraints indicates that the MM portfolio exhibits higher returns and standard deviations than the IM portfolio. In addition, based on the risk-adjusted performance, the MM portfolio still outperforms the IM portfolio. This is evident by the higher Sharpe ratios. The examination of Efficient Frontier points is supporting this. It illustrates that for a given level of risk, the MM portfolio tends to offer higher returns than the IM portfolio. Investors who seek higher returns may find the MM portfolio more attractive.

On the other hand, the IM portfolio provides a lower level of risk for a given return. This is potentially appealing to investors with a lower risk tolerance. The choice between the MM and IM models depends on investor preferences, risk tolerance, and specific objectives. It is important for investors to carefully consider these trade-offs and align their portfolio choices with their unique financial goals and risk preferences.

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