ISSN 2959-6130

Comparison of the Markowitz Model and Index Model in Capital Markets

Chuanze Yang

Abstract:

This paper examines two widely used portfolio optimization models: the Markowitz Model (MM) and the Index Model (IM). We collected and processed historical data on ten stocks for 2001-2021 that included four different sectors, one P500 stock index, and a proxy for a risk-free rate (1-month Fed Funds rate). By computing all properly optimized inputs for the full Markowitz and index models, we find the regions allowing the portfolio under five additional realistic constraints. The study shows that the minimum variance combination and the maximum Sharp ratio are better than the Index Model. In addition, this paper further complements the empirical research of two models and provides some valuable investment suggestions for building the portfolio.

Keywords: Markowitz model, Index model, optimal portfolio, minimal risk portfolio

1. Introduction:

Portfolio selection is one of the most important decisions in financial investment, and a rational investor always seeks to maximize returns with limited risks (Rout, B. & Panda, J. 2019). In portfolio selection studies, the Markowitz model M and I.M. are common methods.

The Markowitz Model was established by the American economist Harry Markowitz in 1952, and it is known as the foundation work of modern portfolio theory. The main idea of this model is to find the portfolio with the least risk given the expected return through the analysis of effective frontier. However, despite its wide application, practical applications have some drawbacks. The Markowitz model is highly sensitive to the input data (expected returns, variances, and covariances). Slight changes in the Markowitz model may lead to significant changes in the final portfolio weight and increase the portfolio instability. In terms of data collection and calculation, the model requires a large amount of historical data to effectively apply the Markowitz model to estimate the expected yield and covariance matrix of assets. Furthermore, the mathematical optimization process required to calculate the optimal portfolio can be overly complex, especially when many assets are included. William shaped the model to address the deficiencies of the MM model and the exponential model was proposed in 1963. In contrast, the I.M. model is simpler and requires less demand for the data.

In this paper, we apply the Markowitz and exponential models to the U.S. stock market and compare and analyze the two models under different constraints.

2. Data

We have collected daily data on total returns for the S&P 500 index from 2001 to 2021 and data for ten individual stocks. These stocks cover four different sectors. Additionally, we have included data for a risk-free instrument (the 1-month Fed Funds rate). To mitigate non-Gaussian effects, we aggregate the daily data into monthly observations. We calculate the monthly return using the following formula:

$$return = \frac{enddateprice}{startdateprice} - 1$$

Table 1 shows the shock ticker symbols for ten companies from different sectors.

Table 1

Ticker symbol	Full name	Sector
NVDA	Nvidia Corporation	Technology
CSGO	Cisco System, Inc.	Technology
INTC	Intel Corporation	Technology
G.S.	The Goldman Sachs Group, Inc.	Financial Services
USB	U.S. Bancorp	Financial Services
T.D. C.N.	The Toronto-Dominion Bank	Financial Services
ALL	The Allstate Corporation	Financial Services
P.G.	The Procter & Gamble Company	Consumer Defensive

JNJ	Johnson & Johnson	Healthcare
CL	Colgate-Palmolive Company	Consumer Defensive

●S&P 500

The S&P 500 (the Standard & Poor's 500) is a stock market index that measures the performance of five hundred of the largest publicly traded companies in the United States. These companies are chosen based on market capitalization, liquidity, and industry representation. The S&P 500 is widely regarded as one of the best indicators of the overall performance of the U.S. stock market. Investors and financial professionals commonly use it as a benchmark for portfolio performance. Figure 1 shows the monthly return of the S&P 500.

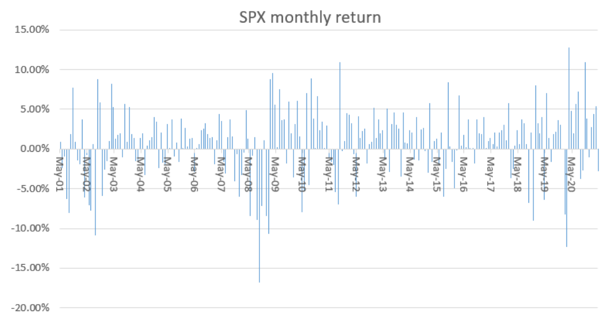

Figure 1

●NVDA

NVDA refers to Nvidia Corporation, an American multinational technology company known for designing and manufacturing graphics processing units (GPUs). Nvidia's GPUs are widely used in the gaming industry, data centers, professional visualization, and artificial intelligence applications. The monthly return of NVDA is shown in Figure 2.

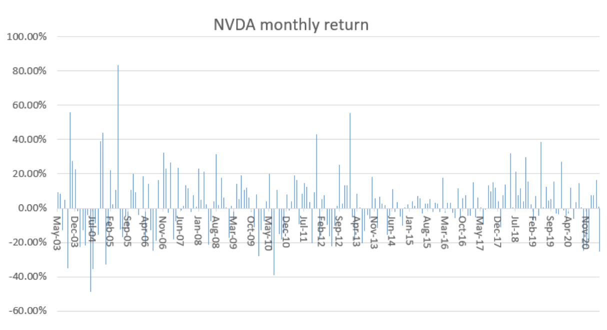

Figure 2

●CSCO

Cisco Systems, Inc. is a headquartered American multinational technology conglomerate. Cisco is a leading provider of networking hardware, software, and services, specializing in designing, manufacturing, and selling networking equipment. Figure 3 shows CSCO's monthly return.

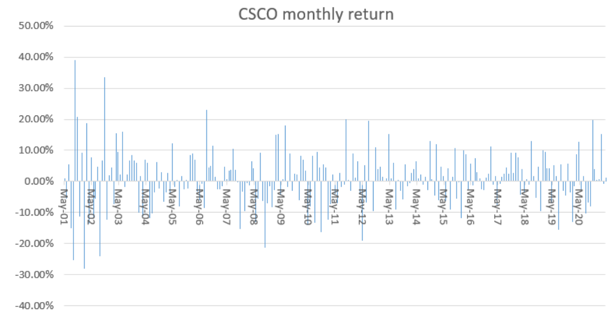

Figure 3

●INTC

Intel Corporation is one of the world's largest semiconductor chip manufacturers. Intel designs and manufactures a wide range of products, including microprocessors, chipsets, graphics processing units (GPUs), and other communication-related hardware. In addition to hardware products, Intel provides software and services related to computing and communications. Figure 4 shows the monthly return of INTC.

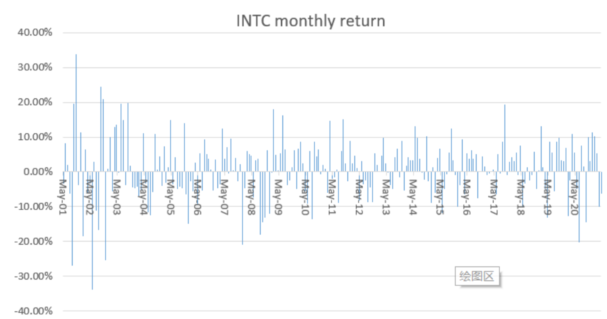

Figure 4

●GS

Goldman Sachs Group, Inc. (G.S.) is a multinational American investment bank and financial services company headquartered in Goldman Sachs. It is one of the world's leading investment banking, securities, and investment management firms. The company provides a wide range of financial services to a substantial, diversified client base. Figure 5 shows the monthly return of G.S.

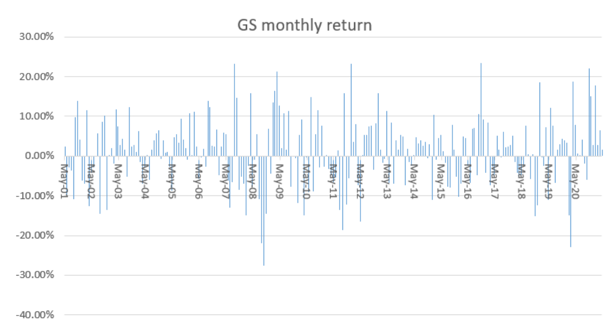

Figure 5

●USB

U.S. Bancorp is an American bank holding company, and it is the parent company of the U.S. Bank National Association. As one of the largest banks in the country, U.S. Bancorp is known for its stability, reliability, and customer-focused approach. It has a strong presence in retail banking, commercial banking, corporate banking, and wealth management, catering to the diverse financial needs of its clients. Figure 6 shows the monthly return of USB.

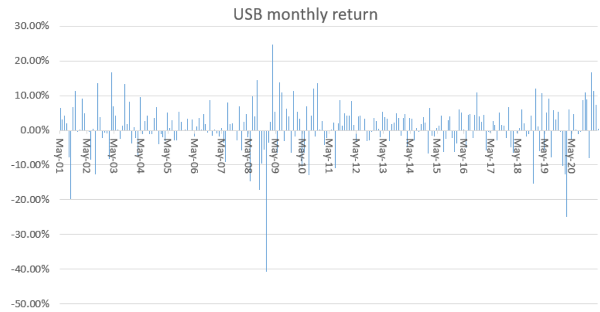

Figure 6

● T.D. C.N.

The Toronto-Dominion Bank is a prominent Canadian multinational banking and financial services corporation. TD Bank was established in 1955 by the merger of the Bank of Toronto and the Dominion Bank. T.D. Bank has become one of Canada's largest banks and a leading financial institution in North America. Figure 7 shows the monthly return of T.D. C.N.

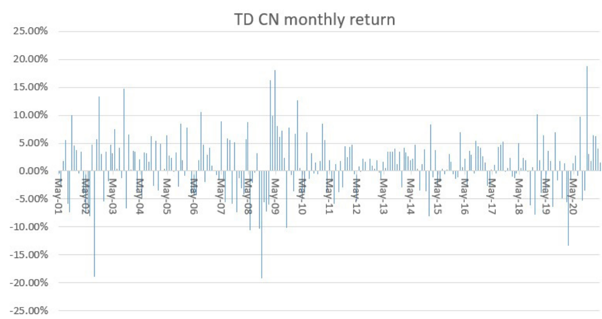

Figure 7

●ALL

The Allstate Corporation is one of the largest insurance companies in the United States, providing insurance products and financial services. In addition to its insurance operations, Allstate has diversified its business through acquisitions and investments in related industries, such as roadside assistance services, home services, and personal finance apps. Figure 8 shows the monthly return of ALL.

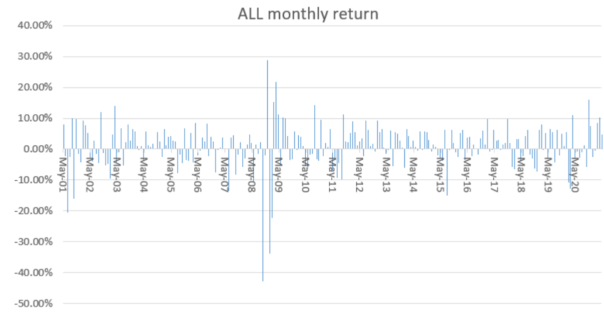

Figure 8

● P.G.

The Procter & Gamble Company, also known as P&G, is one of the biggest producers of daily necessities. P&G operates in many categories: beauty, grooming, health care, fabric, baby, and feminine. P&G's brands are household names across the globe, and the company has a significant presence in over 180 countries. Figure 9 shows the monthly return of P.G.

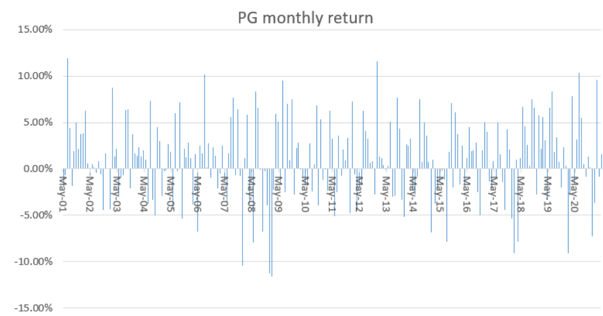

Figure 9

●JNJ

Founded in 1886, Johnson & Johnson is one of the world's most comprehensive and widely distributed healthcare companies, with businesses in three major areas: medical devices, pharmaceuticals, and consumer products. Figure 10 shows the monthly return of JNJ.

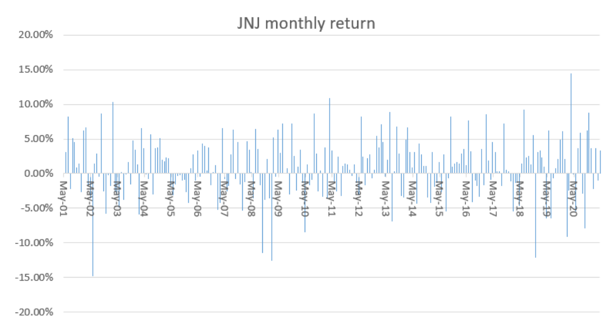

Figure 10

●CL

The Colgate-Palmolive Company is a multinational consumer goods corporation headquartered initially focused on soap and candle manufacturing before expanding into other personal and household care products. Colgate-Palmolive is best known for its oral care products, including toothpaste, toothbrushes, mouthwashes, and dental floss. Figure 11 shows the monthly return of CL.

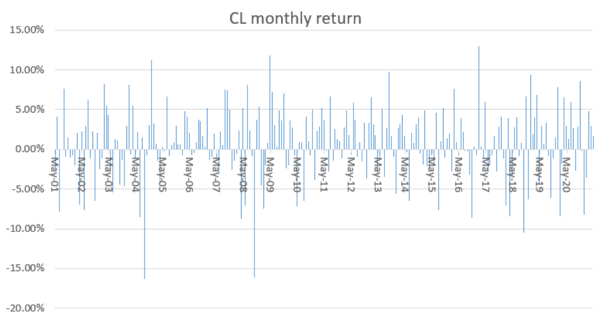

Figure 11

3. Method

In this section, we will introduce two venture capital models in detail: the Markowitz and Index models. The two models will help investors build distinct types of portfolios, such as the minimum variance portfolio and the largest Sharp portfolio.

3.1 Comparison object

Minimum-Variance Frontier: It represents the set of portfolios that offer the lowest possible risk (variance) for a given level of expected return.

$$\begin{cases} \sigma(\vec{w}) \to min(\vec{w}) \\ subjectto: r(\vec{w}) = const \end{cases}$$

Minimal Return Frontier: It represents the lowest expected return an investor will likely receive at each portfolio risk level.

$$\begin{cases} r(\vec{w}) \to min(\vec{w}) \\ subjectto: \sigma(\vec{w}) = const \end{cases}$$

Efficient Frontier: It represents all portfolios that provide the highest expected return at a given risk or the lowest risk at a given expected return.

$$\begin{cases} r(\vec{w}) \to max(\vec{w}) \\ subjectto: \sigma(\vec{w}) = const \end{cases}$$

Minimal Risk Portfolio: The Minimum-Variance frontier:

$$\{\sigma(\vec{w}) \to min(\vec{w})$$

Optimal Risky Portfolio: The tangent point of the efficient frontier and CAL represents the set of portfolios that offer the maximum expected return for a given level of risk or the minimum risk for a given level of expected return.

$$\left\{ \frac{r(\vec{w})}{\sigma(\vec{w})} \to max(\vec{w}) \right.$$

Capital Allocation Line (CAL) represents the portfolio combination of risky and risk-free assets, describing the relationship between the expected return and risk of all possible new portfolios.

Sharpe Ratio: It is the slope of the CAL. We can find the portfolio with a higher Sharpe ratio better. Also, CAL with a higher Sharpe ratio has higher expected returns for a given standard deviation. The formula is as follows:

$$s_p = \frac{E(r_p) - r_f}{\sigma_p}$$

3.2 Markowitz Model

The Markowitz Model, also known as the Mean-Variance Model, is a risk investment model proposed by Harry Markowitz in 1952. The main goal of this model is to minimize the risk (i.e., the variance of return) of a portfolio given a certain expected return. The basic assumption of the Markowitz Model is that investors are risk-averse and choose portfolios based on the expected return and variance of the portfolio. The mathematical expression of the Markowitz Model is as follows:

The expected return of Markowitz Model portfolio P:

$$R_p = \sum_{i=1}^{n} w_i r_i$$

The standard deviation of the Markowitz Model portfolio P:

$$\sigma_P = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j cov(r_i, r_j)}$$

r_i: the expected return on asset i

w_i: the proportion of asset i

n: the number of assets

$cov(\ er, R_j)$: the covariance between the return on asset

i and the return

on asset j

3.3 Index Model

The Index Model is optimized based on the Markowitz Model. The Index Model provides a simpler and more efficient analysis of the portfolio. This model treats the market as a whole and reflects the collective behavior of investors. The Index Model's fundamental assumption is that only one macro factor causes the systematic risk affecting stock returns.

The formula of the Index Model is as follows:

$$r_{it} - r_f = \alpha_i + \beta_i \left(r_{mt} - r_f \right) + \epsilon_{it}$$

r_{it} : the return of stock i
r_f: the risk-free rate
r e: the return of the market portfolio
α_i : excess return
β_i : the sensitiveness of the market return
ϵ_{it} : the residual return
We assume normally that the model distributes with mean zero and standard deviation σ_i .

3.4 constraint

We set five additional constraints that simulate different situations in real economic markets.

1. This constraint is devised to simulate Regulation T by FINRA, which allows broker-dealers to allow their clients to hold 50% or more positions funded by the client's account equity. It can be expressed as follows:

$\sum_{i=1}^{11} |w_i| \leqslant 2$;

2. This constraint is designed to simulate some arbitrary "box" constraints on weights, which the customers may provide. The expression:

$|w_i| \leqslant 1$, for $\forall i$;

3. Without additional optimization constraints, it represents how the area of allowable portfolios in general and the efficient frontier in particular look if there are no constraints.

4. This constraint is designed to simulate the typical limitations of the American mutual fund industry: an American open-ended mutual fund cannot have any short position. The expression:

$w_i \geqslant 0$, for $\forall i$;

5. In this constraint, we include the broad index in the portfolio to determine whether it will have a positive or negative impact. The expression:

$w_1 = 0$

4. Data analysis

The Solver Table is a powerful computational tool to determine the permissible portfolio regions within the Markowitz Model and the Index Model. We construct these portfolios with a diverse set of constraints. Specifically, we explore and calculate three crucial frontiers: the efficient frontier, the inefficient frontier, and the minimum variance frontier. During this analysis, we identify two pivotal points on the efficient frontier: the Global Minimal Variance point and the Maximal Sharpe point, also known as the Efficient Risky Portfolio. These points are paramount in understanding portfolio characteristics and optimizing investment decisions. Additionally, we establish the Capital Allocation Line, a critical component that enables a comparative examination of the differences between the MM and I.M. models. By conducting this comprehensive analysis through Solver Table, we understand the permissible portfolio regions within each model, their associated risk-return profiles, and the nuanced distinctions between the two models, thus empowering us to make informed investment choices.

Based on monthly data, we use the Solver Table to calculate the required estimates for each stock, and the result is shown below:

Table 12

	SPX	NVDA	CSGO	INTC	BAC	USB	TDCN	ALL	PG	JNJ	CL
Annual Average Return	7.5%	32.8%	9.7%	8.9%	10.8%	9.9%	11.0%	10.1%	9.4%	8.5%	7.1%
Annual StDev	14.9%	55.8%	30.8%	30.5%	29.6%	23.7%	18.1%	24.9%	14.6%	14.8%	15.3%
beta	1.00	1.98	1.32	1.19	1.41	0.97	0.79	1.06	0.41	0.54	0.45
alpha	0.00	0.18	0.00	0.00	0.00	0.03	0.05	0.02	0.06	0.04	0.04
residual Stdev	0.0%	47.4%	23.8%	24.9%	20.9%	18.8%	13.9%	19.3%	13.3%	12.4%	13.8%

4.1 Correlation test

The formula for the correlation coefficient of two stocks:

$$\rho(X,Y) = \frac{cov(X,Y)}{\sigma_X \cdot \sigma_Y}$$

Table 13 below shows the correlation of the ten stocks. Chart 13 shows that the correlation of stocks belonging to the same industry is strong; for example, the correlation coefficient of INTC and CSGO in the technology industry is strong. At the same time, the correlation of stocks in different industries is weak.

Table 13 Correlation of stocks

	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL
SPX	100.0%	52.7%	63.7%	57.8%	70.8%	60.9%	64.5%	63.0%	41.2%	54.2%	44.0%
NVDA	52.7%	100.0%	48.7%	52.4%	34.3%	16.0%	33.8%	15.7%	6.0%	16.5%	6.9%
CSCO	63.7%	48.7%	100.0%	61.4%	48.7%	32.8%	41.0%	29.7%	22.0%	23.9%	16.5%
INTC	57.8%	52.4%	61.4%	100.0%	41.1%	28.0%	41.2%	28.6%	13.6%	32.5%	11.0%
GS	70.8%	34.3%	48.7%	41.1%	100.0%	47.2%	49.4%	41.7%	17.3%	29.6%	20.3%
USB	60.9%	16.0%	32.8%	28.0%	47.2%	100.0%	53.9%	54.0%	33.6%	23.4%	21.8%
TD CN	64.5%	33.8%	41.0%	41.2%	49.4%	53.9%	100.0%	41.7%	23.1%	27.3%	21.2%
ALL	63.0%	15.7%	29.7%	28.6%	41.7%	54.0%	41.7%	100.0%	34.6%	45.2%	40.7%
PG	41.2%	6.0%	22.0%	13.6%	17.3%	33.6%	23.1%	34.6%	100.0%	49.4%	48.3%
JNJ	54.2%	16.5%	23.9%	32.5%	29.6%	23.4%	27.3%	45.2%	49.4%	100.0%	52.7%
CL	44.0%	6.9%	16.5%	11.0%	20.3%	21.8%	21.2%	40.7%	48.3%	52.7%	100.0%

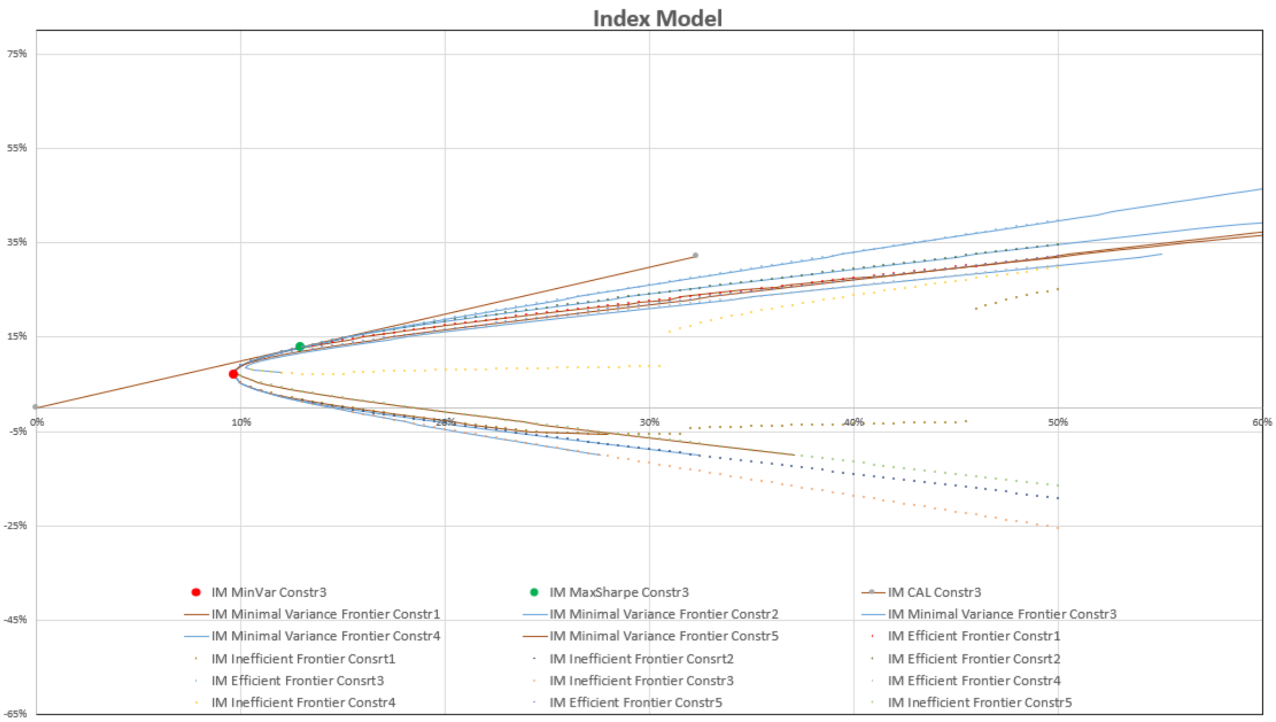

Figure 14

4.2 Empirical analysis

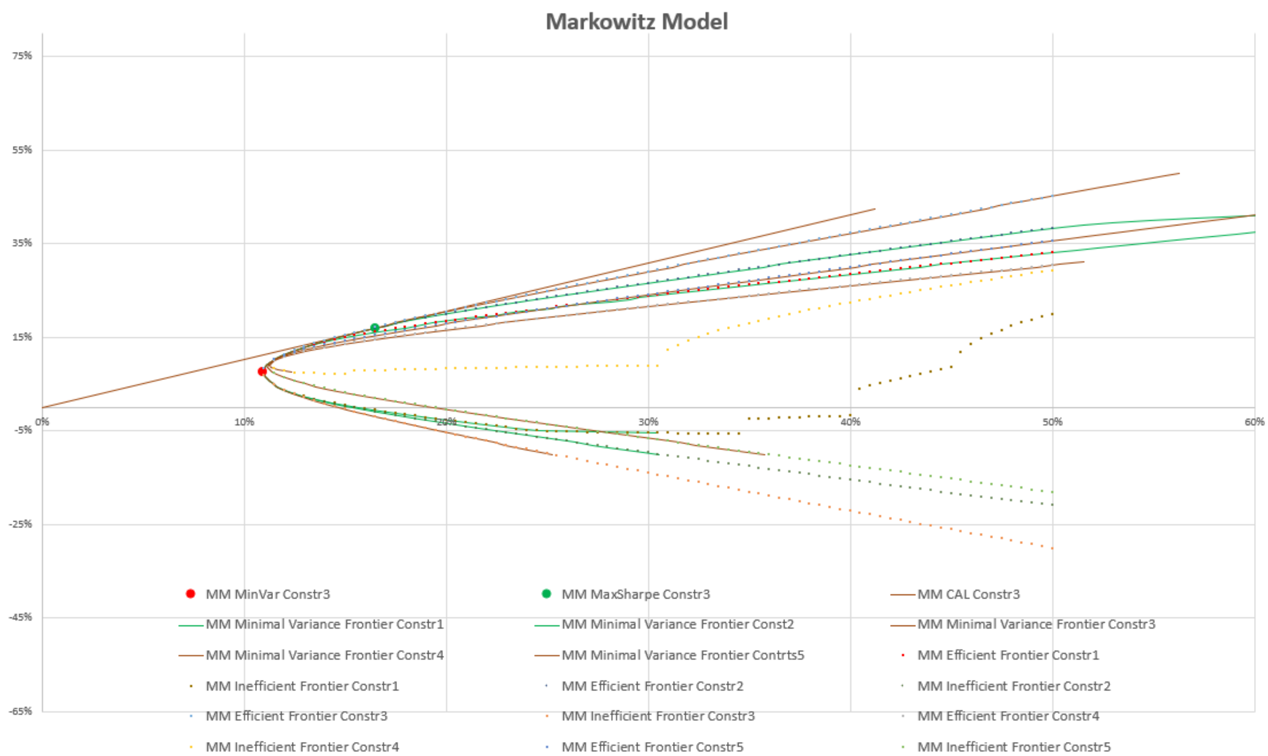

Figure 15

From the above Figure 14 and Figure 15, we can summarize the following points:

1. For the Efficient Frontier, we can achieve more returns for the same risk for constraints two and 3. The point where the efficient frontier is tangent to the CAL is the best choice for the portfolio.

2. For the Inefficient Frontier, constraints two and constraints 3 achieve smaller returns. For the same level of risk, rational investors will not choose the points on the Inefficient Frontier to invest in.

3. The point on the far left of the Minimum Variance Frontier is the smallest variance of portfolios and is called the global minimum variance portfolio. The point where the Minimum Variance Frontier is tangent to the CAL is the maximum Sharp point.

4.3 Comparison

Table 16

MM (Constr1):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	38.37%	-2.97%	-2.89%	1.33%	-5.90%	-0.30%	19.41%	-11.48%	25.93%	18.83%	19.67%	7.51%	10.95%	0.685	CAL:	0.0%	0.0%
MaxSharpe	-42.74%	15.75%	-1.15%	-6.11%	3.25%	6.48%	35.29%	1.07%	45.71%	30.00%	12.45%	14.01%	13.95%	1.004	2.5	35.0%	34.9%
MM (Constr2):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	38.37%	-2.97%	-2.89%	1.33%	-5.90%	-0.30%	19.41%	-11.48%	25.93%	18.83%	19.67%	7.51%	10.95%	0.685	CAL:	0.0%	0.0%
MaxSharpe	-100.00%	21.50%	0.31%	-8.15%	11.46%	12.25%	44.92%	6.87%	52.33%	41.02%	17.48%	16.56%	16.06%	1.031	2.5	41.4%	40.2%
MM (Constr3):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	38.37%	-2.97%	-2.89%	1.33%	-5.90%	-0.30%	19.41%	-11.48%	25.93%	18.83%	19.67%	7.51%	10.95%	0.685	CAL:	0.0%	0.0%
MaxSharpe	-109.97%	22.46%	0.89%	-8.19%	12.73%	13.21%	46.46%	7.90%	53.50%	42.72%	18.30%	16.99%	16.48%	1.031	2.5	42.5%	41.2%
MM (Constr4):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	9.49%	0.00%	0.00%	0.00%	0.00%	0.00%	19.85%	0.00%	28.91%	20.62%	21.13%	8.88%	11.27%	0.788	CAL:	0.0%	0.0%
MaxSharpe	0.00%	10.95%	0.00%	0.00%	0.00%	0.00%	23.73%	0.00%	42.56%	16.17%	6.60%	12.06%	13.12%	0.919	2.5	30.1%	32.8%
MM (Constr5):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	0.00%	-0.97%	0.08%	2.51%	-0.99%	3.50%	24.70%	-8.17%	28.91%	25.58%	24.85%	8.71%	11.18%	0.779	CAL:	0.0%	0.0%
MaxSharpe	0.00%	14.93%	-6.85%	-10.14%	-1.30%	2.43%	30.65%	-2.27%	43.31%	23.61%	5.64%	13.06%	13.69%	0.954	2.5	32.6%	34.2%

IM (Constr1):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	25.62%	-4.04%	-5.27%	-2.81%	-8.73%	0.76%	10.28%	-1.40%	31.27%	27.68%	26.64%	7.15%	9.63%	0.742	CAL:	0.0%	0.0%
MaxSharpe	-47.62%	8.88%	-1.24%	-0.49%	-0.64%	6.67%	29.55%	4.57%	43.95%	33.35%	23.01%	12.07%	12.18%	0.990	2.5	30.2%	30.5%
IM (Constr2):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	25.62%	-4.04%	-5.27%	-2.81%	-8.73%	0.76%	10.28%	-1.40%	31.27%	27.67%	26.64%	7.15%	9.63%	0.742	CAL:	0.0%	0.0%
MaxSharpe	-70.16%	10.32%	-0.57%	-0.11%	0.57%	9.39%	34.25%	7.35%	46.90%	36.94%	25.11%	12.87%	12.92%	0.996	2.5	32.2%	32.3%
IM (Constr3):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	25.62%	-4.04%	-5.27%	-2.81%	-8.73%	0.76%	10.28%	-1.40%	31.27%	27.67%	26.64%	7.15%	9.63%	0.742	CAL:	0.0%	0.0%
MaxSharpe	-70.16%	10.32%	-0.57%	-0.11%	0.57%	9.39%	34.25%	7.35%	46.90%	36.94%	25.11%	12.87%	12.92%	0.996	2.5	32.2%	32.3%
IM (Constr4):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	9.21%	0.00%	33.55%	28.89%	28.35%	8.64%	10.16%	0.850	CAL:	0.0%	0.0%
MaxSharpe	0.00%	6.74%	0.00%	0.00%	0.00%	0.00%	17.75%	0.00%	37.34%	22.75%	15.41%	10.71%	11.72%	0.914	2.5	26.8%	29.3%
IM (Constr5):	SPX	NVDA	CSCO	INTC	GS	USB	TD CN	ALL	PG	JNJ	CL	Return	StDev	Sharpe			
MinVar	0.00%	-3.35%	-3.28%	-1.14%	-6.02%	3.27%	14.25%	1.13%	34.13%	31.54%	29.47%	7.82%	9.75%	0.802	CAL:	0.0%	0.0%
MaxSharpe	0.00%	7.68%	-5.84%	-4.45%	-6.81%	2.56%	22.82%	0.47%	38.76%	26.69%	18.11%	10.84%	11.48%	0.944	2.5	27.1%	28.7%

Table 16 shows the weights and results of the portfolios constructed by the Markowitz Model and Index Model with minimum variance and maximum Sharp ratio under five constraints.

We can find that the Sharpe ratio of the Index Model is always greater than that of the Markowitz Model for the comparison under the smallest variance. In other words, the outcome for I.M. in the case of limited variance is better.

As can be seen from the table above, the maximum Sharpe of the Markowitz Model is greater than the maximum Sharpe of the Index Model under constraints 1, 2, 3, and 5. In this situation, it appears that the Markowitz Model is better. By comparing minimum variance, the return calculation using the Markowitz Model is better than the Index Model under the five constraints. The results of the maximum Sharpe point also indicate that the Markowitz Model has a higher return rate in the maximum Sharpe.

5. Conclusion

This paper introduces two models in modern portfolio theory: the Markowitz Model and the Index Model. We selected ten stocks from four industries to compare and analyze two models. Moreover, we use the excel tool to solve problems under five additional constraints. Finally, this study concludes that the Index Model is a better approximation of the Markowitz Model.