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Topology and Automorphism Structures in General Linear Group

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Abstract:

The general linear group, as a significant topic in algebra and topology, has a wide-ranging research background and application value. With the continuous advancement of mathematical research, the combination of topological structures and automorphism structures has become a key entry point for exploring the properties of the general linear group. This paper investigates the topological properties of general linear group $GL(n,\mathbb{R})$, specifically focusing on compactness, connectedness and the fundamental group, and the automorphism. The first part of the study is dedicated to a detailed analysis of these properties, providing new insights into the topological structural characteristics of $GL(n,\mathbb{R})$. This paper scrutinized the homotopy type within it, giving proof to the fundamental group of $GL(n,\mathbb{R})$, which is showed to be isomorphism to a trivial group. In the second half, This paper introduce and examine the function $\phi(G)=\{A\in GL(n,\mathbb{R}) \mid G\bullet A=G\}$, which is used to identify the automorphism group of a dense subgroup G of $\mathbb{R}^n$. Specifically, the automorphism group of $\mathbb{Q}^n$ is investigated. This paper show that the automorphism group of it is isomorphic to the subgroup $GL(n,\mathbb{Q})$ of general linear group $GL(n,\mathbb{R})$. Through this function, this paper offer an innovative approach to understanding the automorphisms within general linear groups, revealing the deep connections between algebraic and topological aspects of these groups.

Keywords: General linear group; Topological properties; Homotopy; Algebraic structures.

1. Introduction

In the intersection of topology and algebra, this study systematically explores the topological properties of the general linear group, such as connectivity, compactness, and homotopy, while also delving into the structural characteristics of its automorphism group. These properties not only enhance our understanding of the general linear group but also hold significant theoretical and practical implications in fields like modern mathematics and physics. The research of general linear group $GL(n,\mathbb{R})$ has long been playing an important role in algebra and topology, influencing many areas related to mathematics. These groups, composed of all invertible $n\times n$ matrices over the real numbers, possess complex topological properties and automorphism structures. As a result, they remain central to contemporary mathematical research, offering deep insights across a wide range of mathematical disciplines.

For instance, the property of homotopy is crucial to understand. The theorem by Beben and Theriault in 2022 on homotopy fibers demonstrates that comprehending the changes in the homotopy type of a space following a cone attachment is a fundamental challenge in algebraic topology. This is particularly relevant in the context of the LS category, where the impact of such attachments has been extensively studied by researchers such as Kishimoto and Minowa in 2024, Félix and Thomas in 1989, and Hess and Lemaire in 1996, among others. Additionally, the significance of homotopy is further elucidated in essays by Huang and Theriault from 2022 and 2024, which highlight the role of homotopy in the loop space decomposition of manifolds [1-6].

Recent study of Vitalij and Dimitri (2020) demonstrated that subgroups of $GL(n,\mathbb{R})$ are g-reversible for any positive integer n, using the property of automorphism [7]. The concept of g-reversibility extends the notion of reversibility from topological spaces to topological groups, where a group is g-reversible if every continuous automorphism is open. This idea has crucial implications for various classes of groups, including Polish groups and locally compact σ-compact groups, emphasizing the role of automorphisms in preserving topological properties within general linear groups. Additionally, the thesis “Automorphic Representations and L-Functions for GL(N)” by Goldfeld and Jacquet (2021) underlines the importance of automorphism in general linear group, which reinforces the research’s aim to discover the automorphism group of it [8].

These results allow us to visualize how important it is to study these properties of $GL(n,\mathbb{R})$. Therefore, this research seeks to explore the topological properties of $GL(n,\mathbb{R})$, with a particular focus on its connectedness, compactness, and homotopy type. In addition, by examining the automorphism structures, this paper can gain more understanding into how these groups interact with various mathematically relevant domains and how their symmetries are expressed in different contexts.

2. Related Lemmas and Proofs

Before starting to study the properties of $GL(n,\mathbb{R})$, This paper need to know some lemmas first, which will facilitate our further understanding and proof.

Lemma 1:

For a subset S of Euclidean space $\mathbb{R}^n$, the following two statements are equivalent:

S is closed and bounded

S is compact, that is, every open cover of S has a finite subcover (Heine-Borel Theory) [9, 10].

Proof:

S is compact ? S is closed and bounded.

Suppose $S \subseteq \mathbb{R}$ is compact.

Take any limit point s of S.

For each $n \in \mathbb{N}$, consider the open interval $(s-\frac{1}{n}, s+\frac{1}{n})$. The collection of these intervals forms an open cover of S. By compactness, there is a finite subcover, meaning x must be in one of these intervals. Thus, $s \in S$, and S contains all its limit points, making it closed.

Consider the collection of open intervals $\{(-r-1, r+1) \mid r \in \mathbb{R}\}$. This collection covers $\mathbb{R}$ and hence covers S. By compactness, there is a finite subcover; hence S must be contained in some union of these intervals, say the union to be $(-r_i-1, r_j+1)$. Therefore, S is bounded.

Now, using cartesian product to construct a subset in higher dimension gives us the property of S^n.

S is closed and bounded ? S is compact.

Let $(s_n)_n$ be a bounded sequence in S. Since every bounded sequence in finite-dimensional space has a convergent subsequence, $(s_n)_n$ has a convergent subsequence $(s_{n_i})_{n_i}$. Denote the limit of $(s_{n_i})_{n_i}$ $\lim_{i\to\infty} s_{n_i} = s$, then $s \in S$ for S is closed. Therefore, the sets of some elements in subsequence can be the finite subcover and makes S compact.

Lemma 2:

The determinant function $det: GL(n,\mathbb{R}) ? \mathbb{R} \setminus \{0\}$ continuous.

Proof:

The formula of determinant is $|A| = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}$, where σ is a permutation of the n entries and, S_n is the set of all the permutations and $sgn(\sigma)$ is the sign of permutation. Set A_{x_0} is a continuous transformation of A_x.

Want to show:

When $x \to x_0, |A_x| \to |A_{x_0}|$.

Consider.

$$A_{x_0} = \begin{bmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{bmatrix},$$

$$|A_{x_0}| = \sum_{\sigma \in S_n} sgn(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)}(x) \tag{1}$$

Because the map $x \mapsto a_{ij}(x)$ is continuous, so their multiplication and addition are continuous too, which means the function is continuous.

Lemma 3:

X is a topology space, X is connected if and only if the locally constant function makes every element in X a constant.

Proof:

Necessity: Assume X is connected.

Consider the map $\phi: X ? S$, where $S = \{0,1\}$.

Create two open sets:

$\{0\} = \{x \in X \mid \phi(x) = 0\}$ *and* $\{1\} = \{x \in X \mid \phi(x) = 1\}$.

$$\{0\} \cap \{1\} = \varnothing, \{0\} \cup \{1\} = X \tag{2}$$

See that the complements of the two sets are the opposites of themselves, so the two sets are both open and closed. However, X is connected which means the only open and closed subset of X is X itself. Without loss of generality, This paper can let $\{0\} = \varnothing$, and now $\{1\} = X$.

Sufficiency: Assume the locally constant function makes every element in X a constant.

Prove by contradiction. If X is not connected, then there exist two open subsets of X, namely A, B, where $A \cap B = \varnothing, A \cup B = X$. Consider the map ϕ and $a \in A, b \in B \Rightarrow \phi_A(a) = m, \phi_B(b) = n.$ It is easy to confirm the map is a locally constant function but leads to a contradiction for the two subsets have two different constant

images.
Lemma 4:
The fundamental groups of two homeomorphic spaces are isomorphic.
Proof:
Let X,Y to be two homeomorphic spaces with a map $F: X ? Y$. Consider a map.

$$f:\pi_1(X,x_0)\to\pi_1(Y,y_0) \tag{3}$$

Such that for a path γ_x in X based at x_0, $f(\gamma_x)$ is a path in Y which takes $y_0=F(x_0)$ as the base. To check the map is well-defined, let γ_{x_1}, γ_{x_2} to be two homotopic paths in X. Since F is a continuous map, the images must be homotopic too. For two elements $[\gamma_{x_1}]$ and $[\gamma_{x_2}]$ in $\pi_1(X,x_0)$.

$$\begin{gathered} f([\gamma_{x_1}]\circ[\gamma_{x_2}])=f([\gamma_{x_1}\circ\gamma_{x_2}])=[F\circ_F(\gamma_{x_1}\circ\gamma_{x_2})]= \\ [(F\circ_F\gamma_{x_1})\circ(F\circ_F\gamma_{x_2})]=[(F\circ_F\gamma_{x_1})][(F\circ_F\gamma_{x_2})]= \\ f([\gamma_{x_1}])\circ f([\gamma_{x_2}]) \end{gathered} \tag{4}$$

This shows that the map f is a homomorphism.
Injective:
If $f([\gamma_{x_1}])=[(F\circ_F\gamma_{x_1})]$ is a trivial element, then γ_{x_1} is, by definition, an identity path and [γ_{x_1}] is the identity.
Surjective:
For some $[\gamma_y]$ in Y, let γ_y to be a representative, such that $\gamma_y:[0,1]\to Y$. Because F is a homeomorphism, there exists $\gamma_x:[0,1]\to X$ for all γ_y such that $F\circ_F\gamma_x=\gamma_y$.
This paper see f can be an isomorphism between the fundamental groups of two homeomorphic spaces.
Lemma 5:
Every dense subset A of $\mathbb{R}^n$ contains n vectors that form a basis of $\mathbb{R}^n$.
Proof:
Let $A\subset\mathbb{R}^n$ be a dense subgroup. This paper need to show that A contains n vectors that are linear independent, which will form a basis for $\mathbb{R}^n$.
Since A is dense in $\mathbb{R}^n$, there exists a vector $v_1\in A$ such that $v_1\neq 0$. This is because the only vector that cannot be approximated arbitrarily closely by non-zero vectors is the zero vector itself, and A being dense implies it contains non-zero vectors.
Assume k linear independent vectors are chosen to be in A, and consider a set of all the vectors in $\mathbb{R}^n$ that can be noted as $x_1v_1+x_2v_2+\cdots+x_kv_k$. The set is a subspace of $\mathbb{R}^n$ of dimension k. Therefore, $\{v_1,v_2,\cdots,v_k\}$ cannot cover $\mathbb{R}^n$. Since A is dense in $\mathbb{R}^n$, there exist some vector $v_{k+1}\notin\{v_1,v_2,\cdots,v_k\}$ such that v_{k+1} is arbitrarily close to some vector which is not in the span either. Continuing this construction gives us a final basis $\{v_1,v_2,\cdots,v_n\}$ of $\mathbb{R}^n$.
Lemma 6:
Let G be a divisible subgroup of $\mathbb{R}$, then the following holds:

$$(\mathbb{Q}\setminus\{0\})\cdot\phi(G)=\phi(G) \tag{5}$$

$(\mathbb{Q}\setminus\{0\})\cdot I\subseteq\phi(G)$ (I is the identity of $\phi(G)$).
If $\gamma\circ\delta=\delta\circ\gamma$ for all $\gamma\in\phi(G)$, where $\gamma,\delta\in\phi(G)$, then $\phi(G)\subseteq\phi(\delta(G))$.

$$\phi(G)=\{x\in\mathbb{R}\setminus\{0\}\mid x\cdot G=G\} \tag{6}$$

Proof:
Since G is set to be divisible, the equation $(\mathbb{Q}\setminus\{0\})\cdot G=G$ holds. Suppose $\psi_A\in\phi(G)$ and $q\in\mathbb{Q}\setminus\{0\}$, then $q\cdot\psi_A(G)=\psi_A(q\cdot G)=\psi_A(G)=G$. Therefore, by definition, $q\cdot\psi_A\in\phi(G)$. And this gives to $(\mathbb{Q}\setminus\{0\})\cdot\phi(G)=\phi(G)$.
Since I is the identity of $\phi(G)$, this follows from a).
For $\gamma,\delta\in\phi(G)$,
$\gamma(\delta(G))=\gamma\circ\delta(G)=\delta\circ\gamma(G)=\delta(\gamma(G))=\delta(G)$.
That is $\gamma\in\phi(\delta(G))$. Because γ is chosen arbitrarily, $\phi(G)\subseteq\phi(\delta(G))$ holds.
This is clear once we let $x=A$ to be a one entry matrix.
3. Topological Properties of $GL(n,\mathbb{R})$

3.1 Compactness

Relatively, compactness for the general linear group is easier to discover by Lemma 1, This paper will show property first.

3.1.1 Non-compactness of $GL(n,\mathbb{R})$

Proof:
By Lemma 1, one way to check is $GL(n,\mathbb{R})$ is compact is to discuss if it is closed and bounded at the same time. Consider a diagonal matrix.

$$A=diag\left(x,\frac{1}{x},1,\cdots,1\right)_{n\times n} \tag{7}$$

When x goes infinite, the determinant of A goes infinite too. So, the elements in $GL(n,\mathbb{R})$ cannot be contained in a bounded set, which means the group is not bounded.

3.1.2 Proposition: One compact subgroup of $GL(n,\mathbb{R})-O(n)$

Proof:

As we know, every element $O?O(n)$ satisfies $O^TO=I$, where O^T is the transpose matrix. O is called the orthogonal matrix consisting of identical column vectors. As this defined, $¡¬O¡¬=n$, which means $O(n)$ is bounded by the map $|\ |*|\ |:\mathbb{R}^n?\mathbb{R}$.

Next, what is left needs to be done is to prove that the set is closed. Since elements in the set is bounded, This paper can consider a convergent sequence $(O_x)_x$ with the limit of the sequence O_k being outside of the set, then there must exist $\epsilon>0$, such that $B(O_k,\epsilon)\cap O(n)=\varnothing$. However, for all $\epsilon>0$, there exists $N_\epsilon>0$, such that for $x\geqslant N_?$, $O_x\in B(O_k,\epsilon)$, and this leads to a contradiction. Therefore $O(n)$ is a quasi-compact set. Because $O(n)$ is Hausdorff space, it is compact space.

3.2 Connectedness

3.2.1 Proposition: The subgroup $SL(n,\mathbb{R})$ of $GL(n,\mathbb{R})$ is connected

Proof:

Define a map $\beta:SL(n,\mathbb{R})?\Lambda$, where the discrete topology is given to $\Lambda=\{1\}$. This can also be defined as the determinant function since all the determinants of the matrices in $SL(n,\mathbb{R})$ are equal to one. It is not hard to see $SL(n,\mathbb{R})$ is connected by Lemma 3.

3.2.2 Two connected components of $GL(n,\mathbb{R})$

Proof:

Say $GL(n,\mathbb{R})$ has two connected components. This paper will give proof to why it contains two connected components latert.

Consider a diagonal matrix $D=diag(d_1,d_2,\cdots,d_n)$ and a map.

$$h:GL(n,\mathbb{R})\to SL(n,\mathbb{R}) \tag{8}$$

$$AD\to B \tag{9}$$

Here one must prove the map works. As we know, the matrix in $SL(n,\mathbb{R})$ has a determinant of one. This paper just need to make $|AD|=1$. Because $|AD|=|A|\times|D|$, $|A|\neq 0, |D|=\prod_{i=1}^{n} d_i$, This paper can divide $|A|$ at both sides and get $\prod_{i=1}^{n} d_i=\frac{1}{|A|}$. One simple way to derive the matrix D is to let all the ds have the same value, that is $d=\left(\frac{1}{|A|}\right)^{\frac{1}{n}}$ and a little bit more change when the $|A|$ changes between positive and negative (when $|A|<0$, This paper need to put a negative sign before d). This paper has a map that maps all the elements in $GL(n,\mathbb{R})$ to $SL(n,\mathbb{R})$, so $GL(n,\mathbb{R})$ is connected depending on different components.

It is time to prove it has two connected components, namely X and Y. Consider a path.

$$p:[0,1]\to GL(n,\mathbb{R}) \tag{10}$$

Such that $p(0)=X$, $p(1)=Y$. And consider a composite function.

$$f(t):=det(p(t)) \tag{11}$$

By Lemma 2, $f(t)$ is a continuous function too. If for $\forall x\in X, y\in Y$, their determinants have different sign, then $f(t)$ must have different sign too which means it needs to continuously convert from positive to negative or the opposite. This leads to a contradiction because $f(t)$ is continuous and there must exist $f(\alpha)=0$ which is not contained in $GL(n,\mathbb{R})$. At the same time, $GL(n,\mathbb{R})$ is, by definition, not a path-connected space as we cannot find a continuous path between the two connected components.

3.2.3 Proposition: Connected components of $GL(n,\mathbb{R})$ is path-connected

Proof:

Take the one with elements of positive determinants. Consider a map $H:GL(n,\mathbb{R})_+\times[0,1]\to GL(n,\mathbb{R})_+$, such that $H(A,t)=tA+(1-t)I_n$, where $A\in GL(n,\mathbb{R})_+$ and I_n is the identity with positive terminant. It is not difficult to see the path is continuous. For another connected component $GL(n,\mathbb{R})_-$, set the path to be $H(B,t)=tB+(1-t)(-I_n)$, where $B\in GL(n,\mathbb{R})_-$ and $det(-I_n)=(-1)^n$.

3.3 Homotopy

Discussing homotopy within the general linear group, the fundamental group offers a valuable perspective, encompassing concepts like homotopy paths and linear homotopy. This section will explore the fundamental group of the general linear group from two different situations.

3.3.1 The fundamental group of general linear group when $n=1$

For $n=1$, $GL(1,\mathbb{R})=\mathbb{R}'\setminus\{0\}$ can be viewed as two disjoint connected components for one contains positive numbers and the other contains negative, that is $GL(1,\mathbb{R})=\mathbb{R}_-'\cup\mathbb{R}_+'$. Intuitively, $\mathbb{R}_-',\mathbb{R}_+'$ are homeomorphic to the interval $(0,\infty)$ in $\mathbb{R}$. Under proposition 3.2.3, consider a representative path f of an equivalence class of paths $[f]$ in $(0,\infty)$, since the start and end of the path must be the same, f is simply a path stays at the same point, and that implies $\pi_1(\mathbb{R}^+)$ only has identity path as its element. Otherwise, if f moves, there must exists a reentry in the path, which This paper called a discontinuity, which makes f discontinuous. Therefore, the fundamental group $\pi_1(\mathbb{R}^+)$ is equal to $E(trivialgroup)$. By Lemma 4, that is.

$$\pi_1(\mathbb{R}_+')\cong\pi_1(\mathbb{R}^+)=E \tag{12}$$

$$\pi_1(\mathbb{R}_-')\cong\pi_1(\mathbb{R}^-)=E \tag{13}$$

Because these two spaces are disjoint, $GL(1,\mathbb{R})$ can also be written as $\{\mathbb{R}_-',\mathbb{R}_+'\}$. It is easy to see $\pi_1(GL(1,\mathbb{R}),I)=\{[f_{I+}],[f_{I-}]\mid f_{I+}\in\pi_1(\mathbb{R}_-'), f_{I+}\in\pi_1(\mathbb{R}_+')\}$. Hence $\pi_1(GL(1,\mathbb{R}),I)=E$.

3.3.2 The fundamental group of general linear group when $n>1$

Still see the two connected components $GL(n,\mathbb{R})_+$ and $GL(n,\mathbb{R})_+$ independently first. Consider the fundamental group of $GL(n,\mathbb{R})_+$ at base I_n and two paths γ_{I_n} and γ_x, where $\gamma_x(0)=I_n$ and $\gamma_x(1)=I_n$. Consider a map $H:[0,1]\times[0,1]?GL(n,\mathbb{R})_+$, such that $H(A,t)=t\gamma_x(A)+(1-t)\gamma_{I_n}$ for $\forall t,A\in[0,1]$. When $t=0$, $H(A,0)=\gamma_{I_n}$, and this is the identity path sends identity to identity. When $t=1$, $H(A,1)=\gamma_x(A)$, and when A varies, the path becomes a loop based at I_n. Since map H is continuous, γ_x and γ_{I_n} are path-homotopy, which means any path of $GL(n,\mathbb{R})_+$ can be homotopic to the identity path. In conclusion.

$$\pi_1(GL(n,\mathbb{R})_+)=\{[\gamma_{I_n}]\}=\{e\}=E \tag{14}$$

Additionally, $\pi_1(GL(n,\mathbb{R}),I_n)=E.$

4. Automorphism Groups of a Dense Subgroup G of $\mathbb{R}^n$

4.1 New Definition and Identification

To find out the automorphism groups of dense subgroups G of $\mathbb{R}^n$, This section will introduce a new definition that is.

$\phi(G)=\{A\in GL(n,\mathbb{R})\mid G\bullet A=G\}$ where $G\bullet A=\{g\bullet A\mid g\in G\}$.

For the space $\mathbb{R}^n$, it is not difficult to see $\phi(\mathbb{R}^n)=GL(n,\mathbb{R})$. This paper need to show that the automorphism group of G can be identified as the newly defined $\phi(G)$.

Consider a map.

$$f:\phi(G)\to Aut(G) \tag{15}$$

It is necessary to show the map is an isomorphism.

Proof:

Looking back at the definition of $\phi(G)$, the element in the set is defined to keep the group G with right scalar multiplication. Thus, it is direct to induce the map $f(A)=\psi_A$, where ψ_A is an automorphism of G(such that $\psi_A:G?G,\psi_A(g)=g\bullet A$). In another word, A is a map sends G to itself, and it follows a more general denotation of general linear group which is $GL(V)$. For every element in $GL(V)$, it is defined to be a bijection that sends the vector space V to itself. Sometimes, view A as a map denoted as ψ_A for better proof.

To show the map f is well-defined, this paper can consider A to be a map as the element in $GL(V)$, which is bijective and invertible. As we constructed before, the property of homomorphism is easy to get. To check its bijectivity, consider $\psi_A(g_1)=\psi_A(g_2)$, then $g_1\bullet A=g_2\bullet A$, which implies $g_1=g_2$ due to the bijectivity of A. Thus, $\psi_A(g)$ is injective. Similarly, for any $h\in G$, since A^{-1} exists, there exists some $g\in G$ such that $h=g\cdot A$, making $\psi_A(g)$ surjective.

Under this circumstance, this paper are able to tell if the map is an isomorphism.

Homomorphism:

$$f(A\bullet B)=\psi_{A\bullet B}=(g\bullet A)\bullet B=\psi_B\circ\psi_A=f(B)\circ f(A) \tag{16}$$

Where $\circ$ denotes as the composition of the automorphisms in $Aut(G)$.

Injectivity:

Suppose $f(A)=f(B)$ for $A,B\in\phi(G)$, that is $g\cdot A=g\cdot B$ for all $g\in G$. Since the group function is defined to be the scalar multiplication of matrices, A and B must be the same, which means every element in $Aut(G)$ has an inverse image.

Surjectivity:

Let $\psi\in Aut(G)$ be an arbitrary element in $Aut(G)$. Since ψ is an automorphism of G, there exists an $A\in GL(n,\mathbb{R})$ such that $\psi(g)=g\cdot A$ for all g in G. Therefore, for this A have $f(A)=\psi$ which shows f is surjective.

Up to now, it is clear that can construct an isomorphism between $\phi(G)$ and $Aut(G)$, and this paper going to use $\phi(G)$ to identify $Aut(G)$ for contents below.

4.2 Property of the New Definition

For the new definition, a property $|\phi(G)|=|Aut(G)|\leqslant|G|$ is satisfied. In this section later, proof of this property is displayed.

By Lemma 5, suppose a basis $\{v_1,v_2,v_3,\cdots,v_n\}$ of $\mathbb{R}^n$, where $v_n\in G\subseteq\mathbb{R}^n$. For elements in $\phi(G)$, take an arbitrary element ψ_A as an example, $\psi_A(v_n)\in G$ for all $n\geqslant 1$. Therefore, ψ_A is determined by its action on this basis. Consider a map:

$$\varphi:\phi(G)?G^n \tag{17}$$

The map φ is injective because distinct matrices in $\phi(G)$ induce distinct transformations on the basis of $\mathbb{R}^n$ which consists of elements in G. And this gives the conclusion that $|\phi(G)|\leqslant|G^n|$. Since G is dense in $\mathbb{R}^n$ and n is finite, G must be infinite, which means $|G^n|=|G|$. Therefore, $|\phi(G)|\leqslant|G|$, and by the isomorphism created before, $|\phi(G)|=|Aut(G)|\Rightarrow|Aut(G)|\leqslant|G|$.

4.3 The Automorphism Group of $\mathbb{Q}^n$

Recall: A group G is said to be divisible when for every $g\in G$ and every integer $k\neq 0$, there exists $h\in G$ such that $g=k\cdot h$.

that $Aut(\mathbb{Q}^n)\cong\phi(\mathbb{Q}^n)=GL(n,\mathbb{Q})$.

4.3.1 $\phi(\mathbb{Q}^n)\subseteq GL(n,\mathbb{Q})$

Proof:

Consider $A\in\phi(\mathbb{Q}^n)$. By definition, A takes vectors in $\mathbb{Q}^n$ to vectors in $\mathbb{Q}^n$. Let $A=(a_{ij})$, This paper need to show every a_{ij} is rational. Consider the standard basis vectors $e_1,e_2,\cdots,e_i$, where e_i has 1 in the i-th position and 0 elsewhere. Thus, $Ae_i\in\mathbb{Q}^n$ and.

$$Ae_i=\begin{pmatrix}a_{1i}\\ a_{2i}\\ \vdots\\ a_{ni}\end{pmatrix} \tag{18}$$

Since $Ae_i\in\mathbb{Q}^n$, every a_{ji} must be a rational number for all $j=1,2,\cdots,n$. These above show that every a_{ij} of A is rational for multiplying e_i does not change the value of every entry in A. Because A contains entries with rational values and is invertible, $A\in GL(n,\mathbb{Q})$, furthermore, A is chosen arbitrarily, that is $\phi(\mathbb{Q}^n)\subseteq GL(n,\mathbb{Q})$.

4.3.2 $GL(n,\mathbb{Q})\subseteq\phi(\mathbb{Q}^n)$

This way is relatively hard to show, and now must prove a proposition first.

4.3.2 .1 Proposition: For a subgroup H of $\mathbb{R}$, $GL(n,\mathbb{Q})\subseteq\phi(H^n)$ if and only if H is divisible

Proof:

Sufficiency:

Suppose H is divisible.

Let $GL(n,\mathbb{Q})$ $Q=(q_{ij})$, then $q_{ij}\in\mathbb{Q}$. If $q_{ij}=0$, then $H\cdot q_{ij}=\{0\}\subseteq H$. If $q_{ij}\neq 0$, then $q_{ij}\in\mathbb{Q}\setminus\{0\}$. From Lemma 6 b), This paper know that $\mathbb{Q}\setminus\{0\}\subseteq\phi(H)$, and so $q_{ij}\in\phi(H)$. By Lemma 6 d), $q_{ij}\cdot H=H$. Now, construct H^n by using the cartesian product, that is $H^n=H\times H\underset{n}{\overset{?}{\times}}\cdots\times H$, and pick an element h_i in every H such that H^n $h=(h_1,h_2,\cdots,h_i)$. Therefore, $h\cdot Q=\{\sum_{i=1}^{n}h_i\cdot q_{i1},\sum_{i=1}^{n}h_i\cdot q_{i2},\sum_{i=1}^{n}h_i\cdot q_{i3},\cdots,\sum_{i=1}^{n}h_i\cdot q_{in}\}\in H^n$.

That is

$H^n\cdot Q\subseteq H^n$.

Inversely, since $Q^{-1}\in GL(n,\mathbb{Q})$, then.

$$H^n\cdot Q^{-1}\subseteq H^n \tag{19}$$

Multiplying Q at both left sides, this paper get

$H^n \subseteq H^n \cdot Q$. Here conclude that $H^n \cdot Q = H^n$, and, by Lemma 6 d), $Q \in \phi\left(H^n\right)$. Since Q is chosen arbitrarily, $GL(n,\mathbb{Q}) \subseteq \phi\left(H^n\right)$.

Necessity:

Assume $GL(n,\mathbb{Q}) \subseteq \phi\left(H^n\right)$.

Proof by contradiction. If H is not divisible, there exists a non-zero integer k such that $k \cdot H \neq H$. Suppose $A \in GL\left(n,\mathbb{Q}\right)$ such that $A = k \cdot I_n$, $A \cdot H^n \neq H^n$. As proved, $A \notin \phi\left(H^n\right)$.

Now, by proposition 4.3.2.1, as $\mathbb{Q}$ is divisible, $GL(n,\mathbb{Q}) \subseteq \phi\left(\mathbb{Q}^n\right)$. And this gives the final conclusion that $\phi\left(\mathbb{Q}^n\right) = GL(n,\mathbb{Q})$. Up to now, the automorphism group of the dense subgroup $\mathbb{Q}^n$ in $\mathbb{R}^n$ is identified.

1. Conclusion

In this essay, three topological properties and the automorphism structures in general linear group $GL(n,\mathbb{R})$ is showed, and some lemmas are given proof.

In section 3, first, it is stated that $GL(n,\mathbb{R})$ is not compact, but its subgroup $O(n)$ (the orthogonal group) is compact. Then, the two connected components of $GL(n,\mathbb{R})$ are carried out with the additional information that each of the connected component is path-connected. Furthermore, the fundamental group $\pi_1\left(GL\left(n,\mathbb{R}\right), I_n\right)$ is proved to be equal to the trivial group E.

In section 4, one example of automorphism groups of a dense subgroup G of $\mathbb{R}^n$ is chosen to be displayed, which is the automorphism group of $\mathbb{Q}^n$. In the beginning of this section, a new equation is defined such that $\phi\left(G\right) = \{A \in GL(n,\mathbb{R}) \mid G \bullet A = G\}$, where $G \bullet A = \{g \bullet A \mid g \in G\}$. Additionally, an isomorphism is constructed between $\phi\left(G\right)$ and $Aut\left(G\right)$, which is used to identify $Aut\left(G\right)$. As the essay showed, $Aut\left(\mathbb{Q}^n\right) \cong \phi\left(\mathbb{Q}^n\right) = GL(n,\mathbb{Q})$.

The implications of this work extend beyond the immediate findings. The methods and results presented here open new possibilities for research in both algebraic topology and group theory. Future studies could build upon this foundation by exploring more complex generalizations of $\phi\left(G\right)$ or applying these concepts to other groups. In addition, the interaction between topological properties and automorphisms discovered in this study may inspire further research on the classification of topological groups based on automorphic structures. For instance, this concept can be used in other fields where understanding symmetry and structural stability is important. Disciplines such as physics, computer science, and even data analysis are increasingly dependent on the structural continuity of networks and systems and can benefit from the mathematical framework established in this essay. The comprehensive exploration of general linear groups and the automorphisms is not only a theoretical work, but is expected to have practical applications, laying the foundation for innovative methods and interdisciplinary breakthroughs in the foreseeable future.

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