Residue Theorem and Logarithmic Residue Theorem

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Abstract:

This passage is written for researching the application of residue theorem and Logarithmic Residue theorem. The residue theorem is a powerful tool for the analysis of complex functions. It also can be used to calculate the integral of the real functions. Residue theorem is the extension of the Cauchy's integral theorem and Cauchy integral formula. Passage researched two theorems by defining the residue in two ways. Each of them is defined by Laurent series and defined by integral. Then the residue theorem is introduced with its definition and applications. The passage points out two instances for the applications of the residue theorem. After the residue theorem, the Logarithmic Residue theorem is listed with its definition and applications. There are also three instances for its applications. The residue theorem and Logarithmic Residue theorem mainly contribute for the research of the complex functions' integral calculations. The residue theorem provides powerful mathematical tools in certain special types of real integration problems.

Keywords: Residue, Residue Theorem; Logarithmic Residue Theorem; Complex Function

1. Introduction

This passage introduces the residue theorem and the Logarithmic Residue theorem. Residue theorem uses the formular to find the integral of f(z) by finding the sum of the singularities in the simple closed curve and timing it with $2\pi i$. In Logarithmic Residue theorem, it proves the argument principle and provides an efficient way to figure out the number of null points.

In 2018, Zhu and Li did research on the similarities and differences between the complex function and the real function to help students master the complex functions better [1]. In 2024, Chi did research on solving several integral problems in which integral path has singularities in it by using the Composite closed-circuit theorem, Cauchy integral formula and residue theorem [2]. In 2023, Wang, Chen, Wang et al. did research on real variable definite integral problem which is unavailable to solve in Quantum Mechanics and Solid-State Physics by constructing the integral contour and using the residue theorem in complex function [3]. In 2023, Zhang, Du and Ma did research on extension for a type of generalized integral by using the residue theorem [4]. In 2023, Jiang did research on different solution for complex integral of the same closed curve with parametric form of definite integral, Cauchy integral theorem, Cauchy integral formula, Laurent series, residue theorem, logarithmic residue method [5]. In 2022, Zhou and Huang did research on solving two types of infinite integral by using residue theorem [6]. In 2010, Zhang did research on the transforming the calculation problem of real integral into the calculation of residue by using residue theorem to create an integrand [7]. In 2020, Shen, Shao, Sun et al. did research on Time Domain Electromagnetic Field Numerical Integration Calculation System through residue theorem [8].

In this passage, readers will have a deeper understanding about the residue theorem and the Logarithmic Residue theorem by knowing their definitions and applications.

2. Residue Theorem and Logarithmic Residue Theorem

2.1 Residue Theorem

In the daily calculations of the improper integral, it will be difficult for us to use the methods of calculating the improper integral. Therefore, the residue theorem can make it easier.

In order to introduce the residue theorem, what are residues needed to be figured out first. There are two definitions of residues.

Theorem 1. [9] z_0 is a isolated singularity of $f(z)$. In the neighborhood $0<|z-z_0|<R$ of z_0, $f(z)=\ldots+C_{-n}(z-z_0)^{-n}+\ldots C_{-1}(z-z_0)^{-1}+C_0+C_1(z-z_0)+\ldots+C_n(z-z_0)^n$, with C_{-1} is the residue of $(f(z),z_0)$. It is written as $\operatorname{Res}(f(z),z_0)$ or $\underset{z=z_0}{Res} f(z)$.

Theorem 2. In the neighborhood $0<|z-z_0|<R$, there has a simple closed curve C with point z_0 inside. $\operatorname{Res}(f(z),z_0)$ is equal to the integral

of curve C divided by $2\pi i$. It can be written as $Res\left[f(z),z_0\right]=\frac{1}{2\pi i}\oint_c f(z)dz$. Pay attention to the curve C since it's a random curve in the deleted neighborhood and it must enclose the point z_0.

After introducing the definition of the residues, next step is introducing the residue theorem. Let $f(z)$ be analytic in the area D except for some isolated singularities, and C is a simple closed curve which encloses all the singularities in the area D, so $\oint_c f(z)dz=2\pi i\sum_{k=1}^{n}ReS\left[f(z),z_k\right]$. As the second definition of residue and the residue theorem are observed, it shows out that the residue theorem is a special situation for the second definition of the residue. In addition, the equation can also be expressed as to find the integral of the $f(z)$, which is finding the sum of all the singularities in the curve and time with $2\pi i$.

Here are two examples.

Example 1. Calculate $\oint_{|z|=2}\frac{5z-2}{z(z-1)^2}dz$.

According to the residue theorem,

LHS= $2\pi i\sum_{k=1}^{n}Res\left[\frac{5z-2}{z(z-1)^2},zk\right]$. Therefore, the problem of calculating the integral can be transformed into calculate the residue. There are two steps.

1.Figure out the number of the isolated singularities.

The singularities are the points which are not analytic for $f(z)$. Two point can be figured out easily which are $z=0$ and $z=1$.

2.Find out the residues.

After judging if they are removeable singularity, vertex, essential singularity with limit,

$z=0$ is a vertex for first-order pole, so

$Res\left[f(z),0\right]=\lim_{z\to 0}(z-0)\frac{5z-2}{z(z-1)^2}=-2$;

$z=1$ is a vertex for second-order pole,

$Res\left[f(z),1\right]=\frac{1}{(z-1)!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{5z-2}{z(z-1)^2}\right]=$.

$\lim_{x\to 1}\left(\frac{5z-2}{z}\right)'=\lim_{x\to 1}\frac{2}{z^2}=2$

Therefore,

$$\oint_{|z|=2}\frac{5z-2}{z(z-1)^2}dz=2\pi i(-2+2)=0 \tag{1}$$

Example 2. Calculate $\oint_c\frac{e^z}{z(z-1)^2}dz \quad C:|z|=2$.

1.Figure out the number of the isolated singularities.

The singularities are the points which are not analytic for $f(z)$. So, these two point can be found out easily which are $z=0$ and $z=1$.

2.Find out the residues.

After judging if they are removeable singularity, vertex, essential singularity with limit,

$z=0$ is a vertex for first-order pole,

$Res\left[f(z),0\right]=\lim_{z\to 0}z\frac{e^z}{z(z-1)^2}=\lim_{z\to 0}\frac{e^z}{(z-1)^2}=1.$

$z=1$ is a vertex for second-order pole,

$Res\left[f(z),1\right]=\frac{1}{(z-1)!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{e^z}{z(z-1)^2}\right]=$.

$\lim_{z\to 1}\frac{e^z\cdot z-e^z}{z^2}=0$

Therefore,

$$\oint_c\frac{e^z}{z(z-1)^2}dz=2\pi i(1+0)=2\pi i \tag{2}$$

2.2 Logarithmic Residue Theorem

After researching the residue theorem, Logarithmic Residue theorem will be introduced in the rest pages.

Theorem 3.[10] Logarithmic can be defined as an integral $\frac{1}{2\pi i}\int_C\frac{f'(z)}{f(z)}dz$. C is a simple closed curve and $f(z)$ is analytic in C, analytic on C but not equal to Zero. Logarithmic Residue theorem can prove argument principle and provide an efficient way to figure out the amount of null points.

Therefore,

$$\frac{1}{2\pi i}\int_C\frac{f'(z)}{f(z)}dz=N(f,C)-P(f,C) \tag{3}$$

In the equation, $N(f,C)$ and $P(f,C)$ each represent the null point and the vertex pole in the C.

Theorem 4. (The extension of Logarithmic Residue theorem) Let C be a Contour, and $f(z)$ fit the following conditions:

(1) $f(z)$ is meromorphy in the C.

(2) $f(z)$ is analytic on the C and doesn't equal to zero.

(3) $\phi(z)$ is analytic on C and in the C.

$f(z)$ has null points at $a_1,a_2,\ldots,a_k$, and they have order $a_1,a_2,\ldots,a_k$. $f(z)$ has vertex pole $b_1,b_2,\ldots b_m$, and they have order $\beta_1,\beta_2,\ldots\beta_m$. It holds that

$$\frac{1}{2\pi i}\int_c \phi(z)\frac{f'(z)}{f(z)}dz=\sum_{i=1}^{k}a_i\phi(a_i)-\sum_{i=1}^{m}\beta_i\phi(b_j) \tag{4}$$

Next four examples will be given by Logarithmic Residue theorem.

Example 3. Calculate $\int_{|x|=2}\frac{z^5}{z^3-1}dz.$

Since $\frac{z^5}{z^3+1}=\frac{z^2}{z^3+1}\cdot z^3$, let $f(z)=(z^3+1)^{\frac{1}{3}}$, $\phi(z)=z^3$.

$\frac{f'(z)}{f(z)}=\frac{z^2}{z^3+1}$ in $|z|=2$, $f(z)$ has 1/3 order null point -1, $\frac{1+\sqrt{3}i}{2}$, $\frac{1-\sqrt{3}i}{2}$.

Therefore, $\phi(z)=z^3$ is analytic both in and out of the $|z|=2$.

Then,

$$\int_{|x|=2}\frac{z^5}{2^3-1}dz=$$

$$2\pi i\left[\frac{1}{3}\cdot(-1)^3+\frac{1}{3}\cdot\left(\frac{1+\sqrt{3}i}{2}\right)^3+\frac{1}{3}\cdot\left(\frac{1-\sqrt{3}i}{z}\right)^3\right] \tag{5}$$

$$=-2\pi i$$

Example 4. Calculate $\int_{|x|=1}z^n tan\pi z dz$.

Let $f(z)=\frac{1}{cos\pi z}$, $f(z)$ has two first-order vertex pole $\frac{1}{2},-\frac{1}{2}$, so $\frac{f'(z)}{f(z)}=-\frac{\pi sin\pi z}{cos\pi z}$.

Therefore,

$$\int_{|x|=1}z^n tan\pi z dz=$$

$$\int_{|z|=1}\left(\frac{-\pi sin\pi z}{cos\pi z}\right)\left(-\frac{z^\pi}{\pi}\right)dz= \tag{6}$$

$$-2\pi i\cdot\frac{1}{\pi}\left[1\cdot\left(\frac{1}{2}\right)^n+1\cdot\left(-\frac{1}{2}\right)^n\right].$$

Example 5. Calculate the integral $\int_0^{+\infty}\frac{xsinx}{x^2+a^2}dx(a>0)$.

Let F(z) = $\frac{ze^{i\pi}}{z^2+a^2}$. According to $\frac{f'(z)}{f(z)}=\frac{z}{z^2+a^2}$, it turns out that $f(z)=(z^2+a^2)^{-\frac{1}{2}}$.

So $f(z)$ has -1/2 order null point in the up-half plate which is $z=ai$.

Therefore,

$$\int_{-\infty}^{+\infty}\frac{xe^{ix}}{x^2+a^2}dx=2\pi i\cdot\frac{1}{2}\cdot e^{i\cdot ai}=\pi ie^{-a} \tag{7}$$

Separate the real part and the imaginary part:

$$\int_{-\infty}^{+\infty}\frac{xsinx}{x^2+a^2}dx=\pi e^{-a} \tag{8}$$

Then, according to the nature of the even functions:

$$\int_0^{+\infty}\frac{xsinx}{x^2+a^2}dx=\frac{1}{2}\pi e^{-a} \tag{9}$$

Theorem 5. Assume that $f(z)$ is hylomorphism except $a_1,a_2\cdots a_n$ in the up-half plane $\{z\in C: Imz>0\}$ and continuing to the real axis. Therefore, for any $a>0$,

$$\int_{-\infty}^{+\infty}e^{iax}f(x)dx=2\pi i\sum_{k=1}^{n}Res\left(e^{iaz}f(z),a_k\right) \tag{10}$$

Example 6. Calculate Laplace integral $\int_{-\infty}^{+\infty}\frac{cosax}{b^2+x^2}dx.$

Let $f(z)=\frac{1}{b^2+z^2}$. It satisfies the conditions of the theorem above and there is only one first-order vertex pole z=bi

$$Res\left(\frac{1}{b^2+z^2}e^{iaz},bi\right)=\frac{e^{-ab}}{2bi} \tag{11}$$

Therefore,

$$\int_{-\infty}^{-\infty}\frac{cosax}{b^2+x^2}dx=\frac{\pi}{b}e^{-ab} \tag{12}$$

3. Conclusion

According to the passage, the residue theorem and Logarithmic Residue theorem can be used in solving the integral path problems conveniently. Besides, there are also many people did research on these theorems in different aspects. For instances, Quantum Mechanics and Solid-State Physics, Time Domain Electromagnetic Field Numerical Integration Calculation System. In residue theorem, the integral can be calculated by timing the sum of the singularities with 2πi. Logarithmic Residue theorem can calculate the integral by break the original function into a fraction which is a first-order function over its original function. Then the second original function can be found and the null points of the original function can be figured out. Therefore, these two theorems have big influence in the research of the closed curved integral problems. In the future research of residue theorem and Logarithmic Residue, more attentions will be paid on the application of these two theorems.

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