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The Mean Value Theorem and It's Application

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Abstract:

The Mean Value Theorem (MVT) is the central theorem of differential Calculus. It's the major tool to research on function, also bridging the gap between function and derivative. There are lot of researchers focus on it from ancient times until now. This article introduces details about two mean theorems include the Rolle's Theorem and Lagrange Theorem. This paper uses the sample question and refutation to explain the conditions of the theorems, thus addressing the questions that many students will ask when learning definitions of these theorems. The article later examines the application of the MVT. The proof of the unique existence of roots and the inequality is included. The Mean Value Theorem reveals the link between the macroscopic, overall properties for a function on an interval and the microscopic, localized properties of the function at a point. The significance of the application for the mean value theorem is profound, not only promoting the development of mathematical analysis theory, but also playing an important role in multiple fields.

Keywords: Mean Value Theorem; roots; inequality

1. Introduction

The differential Mean Value Theorem (MVT) is divided into Rolle's Mean Value theorem, Lagrange's median theorem and Cauchy's median theorem. The Mean Value Theorem is a complicated theorem which include many different cases such as Rolle's' Theorem. The complete emergence of it has gone through a long process and is the result of joint research of many mathematicians [1-3]. From Fermat's theorem to Cauchy's mean value theorem, it is a process of gradual improvement and continuous forward development. With the continuous improvement of the related mathematical theoretical knowledge, the MVT is also complete, and the method of proof is also diversified.

In 2000, Hu used determinants to extend the first-order differential MVT to higher-order differential MVT [4]. In 2006, Yu provided a new proof for the MVT in differential calculus, then summarized and introduced several generalized forms of the MVT in differential calculus [5]. In 2010, Ding introduced the first and higher-order forms of the MVT and its applications [6]. In 2013, Chen used differential MVT to derive the corresponding Lagrange or Cauchy integral mean value theorem [7]. In 2014, Zhang et al. extended the application of Rolle's theorem to infinite intervals [8]. In 2020, Liang proposed a basic inequality proof method based on the differential MVT [9]. In 2021, Yang and Zhang discussed the application of the MVT in proving equations, inequalities, limits, and roots of equations [10].

The article will explain the conditions of the MVT in a convenient way to understand. The article also includes the examples of MVT application for better understanding of it and how to connect the theorem with other knowledge such as the inequality and the existence of roots.

2. Basic Knowledge

Theorem 1: (Rolle's Theorem) When a function h satisfies following three properties: 1) h is a continuous function in $[a,b]$ which is a closed interval. 2) h is a differentiable function (a,b) which is an open interval. 3) $h(a)=h(b)$. Then the function h will at least have one value of c $\in(a,b)$ with derivative of 0 as shown in Fig. 1. It can write as $h'(c)=0$.

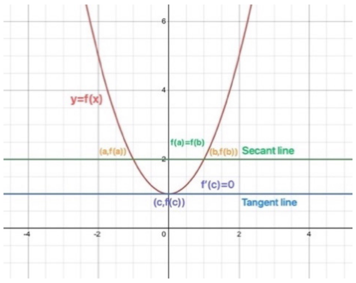

Fig. 1 The graph for Theorem 1

Theorem 2: (Lagrange Theorem) When the function h meets two properties: 1) h is a continuous function in $[m,n] whichisaclosedinterval$. 2) h is differentiable in $(m,n) whichisaopeninterval$. Then the function h will at least have one value of c in the open interval (m,n) that has derivative which is equivalent to the slop for secant line between the two endpoints $(m,h(m))$ and $(n,h(n)) asshownin$ **Fig.**2. In the other words, $h'(c)=\frac{h(n)-h(m)}{n-m}$.

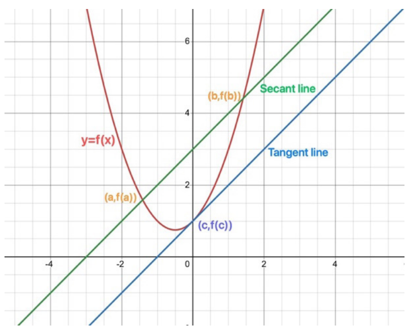

Fig. 2 The graph of Theorem 2

Example 1: Let $h=\sqrt{x-m}$, $x\in[m,n]$. Please find the point $c\in[m,n]$ such that $h'(c)=\frac{h(n)-h(m)}{n-m}$.

Proof: Function h satisfies with two conditions. h is a continuous function on the close interval $[m,n]$; h is differentiable on the open (m,n). $h'=\frac{1}{2\sqrt{x-m}}$, $h(m)=\sqrt{m-m}, h(n)=\sqrt{n-m}$.

Because $h'(c)=\frac{f(n)-f(m)}{n-m}$, $\frac{1}{2\sqrt{c-m}}=\frac{\sqrt{n-m}-0}{n-m}$, $2\sqrt{c-m}=\frac{n-m}{\sqrt{n-m}}, \sqrt{c-m}=\frac{n-m}{2\sqrt{n-m}}, c-m=\left(\frac{n-m}{2\sqrt{n-m}}\right)^2, c=\left(\frac{n-m}{2\sqrt{n-m}}\right)^2+m.$

If a function is differentiable, it must be continuous, but if a function is continuous, it may not be differentiable.

Question: Why the definition of MVT has the condition about the function f is differentiable (a,b) which is a the open interval?

Proof By Contradiction:

Let $f=|x|, x\in[-2,2]$. $\frac{f(2)-f(-2)}{2-(-2)}=0$. However, when $x\in[-2,0), f'(x)=-1$. When $x\in(-2,0], f'(x)=1$. $f'(0)$ doesn't exist.

3. Applications

Example 1: $f(x)$ is continuous in close interval $[m,n]$, and differentiable in open interval (m,n), $f^2(n)-f^2(m)=n^2-m^2$. Proof the function $f(x)f'(x)=x$ at least have one real root in open interval (m,n).

Proof: Based on the known conditions, it's hard to manipulate the function. Now, the author constructs a function that satisfies the Rolle theorem to solve this problem. $f^2(n)-f^2(m)=n^2-m^2$ can be turned into function $f^2(n)-n^2=f^2(m)-m^2$. This shifted function can be associated with the construction function $F(x) whichequalstof(x)-x^2$. Additionally, the function $F(x)$ is a tool to connect Rolle's Theorem to function $f^2(n)-n^2=f^2(m)-m^2$. $f^2(m)-m^2=F(m)$, and $f^2(n)-n^2=F(n)$. Since $F(m)=F(n)$ and $F(x)$ satisfies Rolle's theorem, there is a value x with $x\in(m,n)$, $F'(x)=0$. Perform the derivation of F(c), $F'(x)=2f'(x)(x)-2x$. So, $f'(x)(x)=x$.

Remark 1: The difficulty of this problem is to construct a function that can be connected to the application of Rolle's theorem.

Example 2: $f(x)$ is a continuous function in $[a,b]$ which is a close interval, second-order differentiable in (a,b) which is a close interval, and $f(a)=f(b)=f(c)(a<c<b)$. Prove that there exists $\gamma\in(a,b)$, and $f''(\gamma)=0$.

Proof: Based on the known conditions, we can separate the function in two region and use Rolle's theorem. First, we know that $f(x)$ is also continuous in close intervals $[a,c]$, $[c,b]$, second-order differentiable in open intervals (a,c), (c,b). So, Rolle's theorem leads that there exists $\gamma_1\in(a,c)$ with $f'(\gamma_1)=0$, and $\gamma_1\in(c,b)$ with $f'(\gamma_2)=0$. Because $f(x)$ is second-order differentiable in (a,b) whichisaopenregion. So, $f'(x)$ is continuous in the region $[\gamma_1,\gamma_2]$, and differentiable in open region (γ_1,γ_2). $f'(x)$ in region $[\gamma_1,\gamma_2]$ meets the conditions needed in Rolle's theorem. So, a point $\gamma\in(\gamma_1,\gamma_2)$ must be existed which $f''(\gamma)=0$. Additionally, $\gamma_1\in(a,c),\gamma_1\in(c,b)$ so, $\gamma\in(a,b)$.

Remark: The difficulty of this problem is to associate $f''(\gamma)=0$ to use Rolle's theorem for $f'(x)$. To construct γ_1 and γ_2 by the conditions $f(a)=f(b)=f(c)(a<c<b)$ satisfying Rolle's theorem is hard to construct.

Next, the root uniqueness with Mean Value Theorem will be proved.

Example 3: Function $h(x)$ can be derivable on the closed interval $[0,1]$, $0<h(x)<1$. The derivative of all points in the interval $(0,1)$ hasthecondition $h'(x)\neq -1$. Prove function $h(x)+x-1=0$ has only one root in interval $(0,1)$.

Proof: First step is to prove the equation has real roots with Intermediate Value Theorem.

Let $h(x)+x-1$ bethefunction $g(x)$. Additionally, $0<h(x)<1$ tells us that $g(0)$ must be a negative value ($g(0)=h(0)+0-1=h(0)-1<0$) and $g(1)$ must be a positive value ($g(1)=h(1)+1-1=h(1)>0$). $g(0)$ and $g(1)$ have opposite signs. So, there exists more than one real root c in the interval $(0,1)$ such that $h(c)=0$.

Second step is to prove the equation has only one root with Mean Value Theorem.

Assumption: Function $h(x)+x-1=0$ has two real roots α,β in the interval $(0,1),0<\alpha<\beta<1$. Then $h(\alpha)=1-\alpha$, $h(\beta)=1-\beta$.

Next, the author uses MVT to function $h(x)$ in the interval $[\alpha,\beta]$. So, $h'(c)=\frac{h(\beta)-h(\alpha)}{\beta-\alpha}$,

$h(\beta)-h(\alpha)=h'(c)(\beta-\alpha)$

$h(\alpha)=1-\alpha$, $h(\beta)=1-\beta$ so, $h'(c)=\frac{h(\beta)-h(\alpha)}{\beta-a}=$

$\frac{(1-\beta)-(1-\alpha)}{\beta-\alpha}=-1$, but $h'(x)\neq -1$ contradicts the result of assumption.

Therefore, function $h(x)+x-1=0$ does not have two real roots α,β in the interval $(0,1),0<\alpha<\beta<1$.

In conclusion, function $h(x)$ has only one real root in the interval $(0,1)$.

Example 4: Prove $\frac{x}{1+x}<ln(1+x)<x$, and $x>0$

Proof: Based on the conditions, we can construct a function and an open interval to make it satisfy the MVT to find an intermediate value relationship.

First, the author constructs a function $h(x)=lnx$, in the interval (m,n) with $m=1$, $n=1+x$.

By Mean Value Theorem, there must exist a c in interval (m,n), and $h'(c)=\frac{f(m)-f(n)}{m-n}$.

$h(n)-h(m)=ln(1+x)-ln(1)=ln(1+x)+0=ln(1+x)$,

$n-m=1+x-1=x$. So, $h'(c)=\frac{h(m)-h(n)}{m-n}\rightarrow\frac{ln(1+x)}{x}$

$=h'(c)$

Additionally, the derivative of function $h(x)$ equals to $\frac{1}{x}$

and $h'(c)=\frac{1}{c}$. $\frac{ln(1+x)}{x}=\frac{1}{c}\rightarrow ln(1+x)=\frac{x}{c},m<c<n,$

$1<c<1+x$. Hence, $1<c<1+x\rightarrow\frac{x}{1}<\frac{x}{c}<\frac{x}{1+x}$.

Remark: The difficult point of this question is to construct the point c. Choose the suitable equation to substitute for c, that is using the MVT as connection between two equations to prove the result.

Example 5: Prove $\alpha ln\left(\frac{1+\alpha}{1-\alpha}\right)+cos(\alpha)\geqslant 1+\frac{\alpha^2}{2}$, $(-1<\alpha<1)$.

Proof: Turn $\alpha ln\left(\frac{1+\alpha}{1-\alpha}\right)+cos(\alpha)\geqslant 1+\frac{\alpha^2}{2}$ in to $\alpha ln\left(\frac{1+\alpha}{1-\alpha}\right)+cos(\alpha)-1-\frac{\alpha^2}{2}\geqslant 0$.

Let function $\alpha ln\left(\frac{1+\alpha}{1-\alpha}\right)+cos(\alpha)-1-\frac{\alpha^2}{2},(-1<\alpha<1)$.

It shows that this function is an even function. So, we need to prove $f(\alpha)\geqslant 0$, $0\leqslant\alpha<1$. The positivity or negativity of the derivative can be used to determine the increase or decrease of the original function. $f'(\alpha)=ln\left(\frac{1+\alpha}{1-\alpha}\right)-\alpha-sin(\alpha)+\frac{2\alpha}{2}\geqslant 0$.

It's easy to have $f(0)=0$. Hence, $f(\alpha)\geqslant 0,0<\alpha<1$.

Therefore, $\alpha ln\left(\frac{1+\alpha}{1-\alpha}\right)+cos(\alpha)\geqslant 1+\frac{\alpha^2}{2},(-1<\alpha<1)$.

4. Conclusion

Differential MVT, especially the Lagrange MVT, is a powerful tool for proving inequalities. Inequality problems for functions can be transformed into problems with their derivatives through the Lagrange MVT, so that the properties of derivatives can be used to solve them. The differential median theorem also plays an important role in proving the existence of roots, mainly through the zero theorem and Rolle's theorem. This paper mainly discussed the application of MVT in roots and inequalities. The application of the differential MVT is not limited to prove the existence of roots and inequalities. It can also be used to find limits, especially when dealing with certain types of limit problems. It relates the change in a function to the derivative of a point. In the future, it is hoped that the differential MVT can be applied to more practical problems and promote social progress.

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