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Solving the Small Disturbance Equation and Numerical Method Optimization in Subsonic and Supersonic Flows

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Abstract:

This paper addresses the solution of the Small Disturbance Equation (SDE) under subsonic and supersonic flow conditions using Successive Over-Relaxation (SOR) techniques. The study implements and validates numerical methods tailored to the unique dynamics of these regimes by setting precise boundary conditions, adjusting computational stencils, and managing the Mach number (M_∞) across the solution domain. A detailed comparison of streamline patterns between the regimes illustrates the efficacy of the applied numerical strategies. In the subsonic domain, the flow demonstrates uniform and smooth characteristics, conducive to standard SOR techniques, as evidenced by the rapid decline in residual errors, confirming the method's efficiency for accurate solutions. Conversely, the supersonic regime presents increased complexity where standard finite difference methods encounter notable challenges, necessitating more sophisticated approaches to capture the intricate flow behaviors effectively. Conversely, in the supersonic regime, where the flow behavior exhibits more complex characteristics, the standard finite difference method faces challenges.

Keywords: Small disturbance equation; subsonic flow; successive over-relaxation; computational fluid dynamics.

1. Introduction

The study of fluid dynamics, particularly in the realm of aerospace engineering, is pivotal for understanding the aerodynamic performance of various bodies, such as aircraft wings. Fluid behavior around these structures significantly influences their functionality and efficiency in different flight regimes. Historically, fluid flow analysis under subsonic and supersonic conditions has unveiled distinct physical phenomena that necessitate specialized mathematical models for accurate predictions. The small disturbance equation (SDE), a streamlined version of the full potential equation, assumes minor perturbations in the flow field, making it suitable for analyzing changes in velocity and pressure around airfoils in both subsonic and supersonic speeds [1][2].

Recent advancements in computational fluid dynamics (CFD) have emphasized the importance of linearized CFD methods, particularly the SDE, for their efficiency and precision in predicting unsteady aerodynamics. These methods are crucial in computing generalized aerodynamic forces applicable in aeroelastic analyses [3]. While subsonic flows (M_∞<1) are generally smoother and more predictable, facilitating easier numerical modeling [4], supersonic flows (M_∞>1) present challenges like shock waves and flow separation, complicating the numerical resolution of associated equations [5]. This differentiation in flow characteristics between regimes underscores the necessity for tailored numerical techniques that can adaptively handle the unique aspects of each flow type.

Research Content of This Paper: This paper adopts a numerical approach utilizing the finite difference method paired with the Successive Over-Relaxation (SOR) technique to solve the SDE effectively. The SOR method, an enhancement of the Gauss-Seidel iteration, is particularly adept at solving large linear equation systems derived from the discretization of partial differential equations (PDEs). The technique's ability to adjust the relaxation factor allows for optimization of convergence speeds, crucial for enhancing computational efficiency in CFD simulations [6]. The paper is structured to first elaborate on the mathematical formulation of the SDE and its numerical discretization. It then delves into the distinctive characteristics of subsonic and supersonic flows, followed by a detailed exposition of the methodology employed. Subsequent sections present results from numerical simulations, highlighting comparative analyses of flow patterns and assessing the method's efficacy across both flow regimes. The discussion concludes by reflecting on the findings and their implications for future research in aerodynamic modeling.

2. Small Disturbance Theory

The small disturbance theory simplifies the full potential equations by assuming that the disturbances in the flow variables (such as velocity, pressure, and density) are small relative to their freestream values [7]. This assumption allows the governing equations to be linearized, leading to the SDE, which is much easier to solve numerically while still capturing the essential physics of the problem.
The general form of the SDE can be expressed as:

$$\left(A\frac{\partial\varphi}{\partial x}\right)_x + \left(B\frac{\partial\varphi}{\partial y}\right)_y = 0 \quad (1)$$

Where $\varphi(x, y)$ represents the potential function, and $A(x, y)$ and $B(x, y)$ are coefficients that depend on the flow conditions and the geometry of the object. In subsonic flows, these coefficients remain smooth and continuous, leading to a relatively simple numerical solution. However, in supersonic flows, these coefficients can exhibit discontinuities, corresponding to the presence of shock waves and other complex flow features. In the Python code, the small disturbance equation (SDE) is implemented in a form that assumes the flow is governed by the equation:

$$\left(1-M_\infty{}^2\right)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \quad (2)$$

Where M_∞ is the freestream Mach number. This equation is discretized using a finite difference method, where the coefficient $\left(1-M_\infty{}^2\right)$ multiplies the second derivative of the potential function φ to x, while the second derivative to y remains unaffected. The code solves this discretized form using the SOR method, iterating towards the solution of the potential field $\varphi(x, y)$ under the given flow conditions.

3. Analysis of Flow Characteristics

3.1 Subsonic Flow Characteristics

A flow field is considered subsonic if the Mach number is less than 1 at all points, and it is characterized by smooth streamlines. In subsonic flows, disturbances in the flow field propagate in all directions, and the effects of these disturbances are felt uniformly throughout the fluid [8]. This behavior is reflected in the SDE used in this study, which for subsonic conditions takes the form Eqn.2 with the $M_\infty < 1$, and the coefficient $\left(1-M_\infty{}^2\right)$ remains positive. This ensures that the problem is well-posed and that the solution is stable under standard boundary conditions. The flow characteristics in the subsonic regime are generally predictable, with streamlines that follow the contours of the aerodynamic body smoothly. According to the simulations from Qian (2023), this is evident in the streamline plots, which show gradual changes in the flow direction around the airfoil. The potential function φ varies smoothly across the domain, reflecting the uniformity of the subsonic flow field. The velocity field, derived from the potential function, exhibits continuous gradients, confirming the subsonic nature of the flow [9].

3.2 Supersonic Flow Characteristics

Supersonic flows, where the M_∞ exceeds one, present a distinct set of challenges and characteristics compared to their subsonic counterparts. In this regime, the behavior of the flow becomes markedly more complex, with phenomena such as shock waves, flow separation, and non-linear interactions playing significant roles [10]. These complexities necessitate more sophisticated numerical techniques to accurately capture the flow dynamics. Similarly, based on eqn.2, when $M_\infty > 1$, the coefficient $\left(1-M_\infty{}^2\right)$ becomes negative, indicating a change like the partial differential equation from elliptic to hyperbolic. The numerical solution of the SDE in supersonic regimes requires special considerations to handle the hyperbolic nature of the equation and the associated shock waves. Standard finite difference methods can be unstable or inaccurate if not properly modified for supersonic conditions. One common approach is to use upwind-biased schemes or other forms of numerical dissipation to stabilize the solution and capture the shocks accurately.

4. Methodology

4.1 Mathematical Formulation and Discretization of the SDE

This section provides a comprehensive description of the numerical techniques employed to solve the SDE for both subsonic and supersonic flows. The methodology includes the mathematical formulation of the problem, the discretization approach, boundary conditions, and the detailed implementation of the SOR method in the code. Additionally, specific strategies for optimizing the solution process in different flow regimes are discussed.
Based on the eqn.2, the SDE is discretized using a finite difference method on a uniform grid. The computational domain is divided into $N_x \times N_y$ grid points, where N_x and N_y represent the number of grid points in the x and y directions, respectively. The continuous derivatives in the SDE are approximated using second-order central differ-

ences. The discretized form of the equation at a grid point (i, j) is given by:

$$\left(1-M_\infty{}^2\right)\frac{\varphi_{i+1,j}-2\varphi_{i,j}+\varphi_{i-1,j}}{\Delta x^2}+\frac{\varphi_{i+1,j}-2\varphi_{i,j}+\varphi_{i-1,j}}{\Delta y^2}=0 \quad (3)$$

Where Δx and Δy are the grid spacings in the x and y directions, respectively. This discretization results in a system of linear equations that must be solved to obtain the potential function $\varphi(x, y)$ across the grid. This equation can be rearranged to express the value of $\varphi_{i,j}$ in terms of its neighboring points, which is the discretized equation for the iterative solution process.

$$\varphi_{i,j}=\frac{\dfrac{\varphi_{i+1,j}-2\varphi_{i,j}+\varphi_{i-1,j}}{\Delta x^2}+\dfrac{\varphi_{i+1,j}-2\varphi_{i,j}+\varphi_{i-1,j}}{\Delta y^2}}{2(\dfrac{\left(1-M_\infty{}^2\right)}{\Delta x^2}+\dfrac{1}{\Delta y^2})} \quad (4)$$

4.2 Boundary Conditions

Boundary conditions play a crucial role in ensuring that the numerical solution accurately represents the physical problem. For the left and top boundary, the potential function φ is specified as $\varphi = U_\infty \bullet x$, where U_∞ is the freestream velocity, and x is the position along the boundary. This boundary condition is applied to the first two columns of the grid, representing the inflow where the flow enters the computational domain. For the right boundary (set as outflow), there is no explicit boundary condition. The bottom boundary can be set to a Neumann condition (zero gradient) to simulate a wall or a free-slip condition.

4.3 SOR Method

The basic idea behind the SOR method is to iteratively improve the solution $\varphi(x, y)$ at each grid point by considering the current estimates of the solution at neighboring points. The relaxation factor ω is introduced to adjust the magnitude of the updates, allowing the solution to converge more quickly. According to the equations 3 and 4, the $\varphi_{i,j}$ can be updated by applying the SOR method as:

$$\varphi_{i,j}^{(k+1)}=(1-\omega)\varphi_{i,j}^{(k)}+\frac{\omega}{C_{i,j}}((1-M_\infty{}^2)\frac{\varphi_{i+1,j}^{(k)}+\varphi_{i-1,j}^{(k+1)}}{\Delta x^2}+\frac{\varphi_{i+1,j}^{(k)}+\varphi_{i-1,j}^{(k+1)}}{\Delta y^2}) \quad (5)$$

Where $\varphi_{i,j}^{(k)}$ is the value of the potential function from the previous k-th iteration. ω is the relaxation factor, which controls the weight of the new update, which typically ranges from 1.0 to 2.0, with values greater than 1.0 providing "over-relaxation," and accelerating convergence. $C_{i,j}$ is a normalization factor that ensures the proper weighting of the contributions from neighboring points.

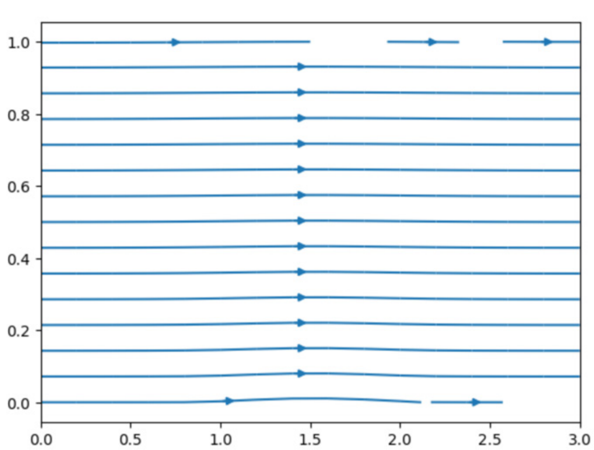
After each complete iteration over the grid, the algorithm checks for convergence by calculating the residual error as:

$$Error=\sum_{i,j}\left|\varphi_{i,j}^{(k+1)}-\varphi_{i,j}^{(k)}\right| \quad (6)$$

If the residual error falls below a predefined tolerance, the iterative process is terminated, and the solution is considered converged.

5. Experimental Results and Discussion

5.1 Subsonic Flow Results

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1900 1.4395502408929943e-11
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Fig. 1 Results of subsonic flow simulations: streamline (left) and residual errors (right) (Photo credit: Original).

$$M_\infty = 0.3 \tag{6}$$

For the subsonic flow simulations, the SOR method demonstrated rapid convergence, with the residual error decreasing exponentially with each iteration. The initial error was relatively small due to the smooth nature of the subsonic flow, and within 2000 iterations, the error fell below the predefined threshold, indicating that the solution had stabilized. The relaxation factor ω was set at 1.5, which was found to optimize the convergence rate without compromising the stability of the solution. This is consistent with the typical range for ω in elliptic problems like the subsonic SDE.

The streamline plot for the subsonic flow is shown in Fig.1. The streamlines are nearly parallel and smooth, indicating a uniform and steady flow. There are slight disturbances near the center of the domain, but these are consistent with the expected behavior of subsonic potential flow around an object. The streamlines closely follow the boundaries of the computational domain, reflecting the influence of the boundary conditions where $\varphi = U_\infty \bullet x$ was specified along the left and top boundaries. The flow remains smooth across the domain, without any significant disruptions, which is typical for subsonic flows where shocks are absent. The convergence behavior and streamline patterns confirm that the numerical method effectively captured the subsonic flow characteristics, providing an accurate solution for the potential field $\varphi(x, y)$.

5.2 Supersonic Flow Results

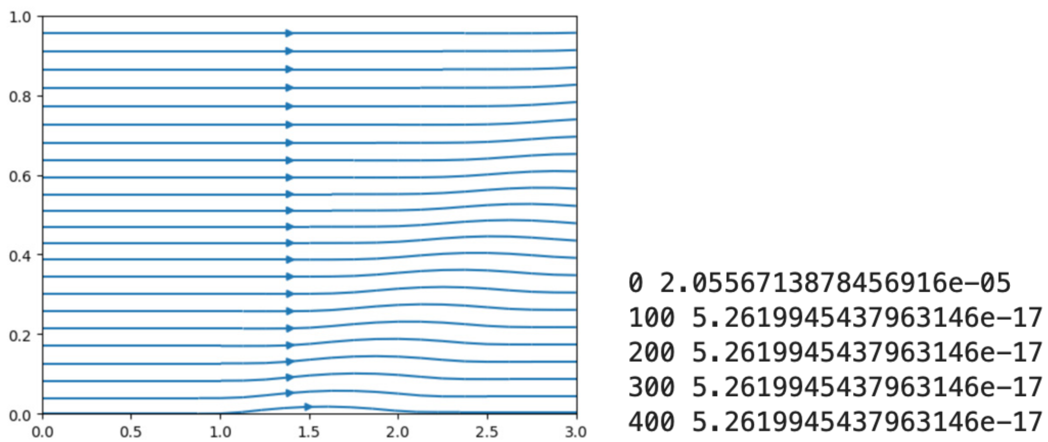

Fig. 2 Results of supersonic flow simulations: streamline (left) and residual errors (right) (Photo credit: Original).

$$M_\infty = 2.5 M_\infty = 0.3 \tag{7}$$

In the supersonic flow case, the SOR method's convergence exhibited different characteristics due to the hyperbolic nature of the equation and the presence of shock waves. The initial residual error was 2.05×10^{-5}, which quickly dropped to 5.26×10^{-17} within 400 iterations. Similar to the subsonic case, the relaxation factor ω was again set to 1.5, and this value was found to be suitable for maintaining stability while achieving fast convergence. However, unlike the subsonic case, the error decreased to a near-zero value much earlier, around 300 iterations, indicating that the solution had reached a steady state quickly.

The streamline plot for the supersonic flow shows more complex behavior compared to the subsonic case (shown in Fig.2). The streamlines exhibit slight curvature, especially near the center of the domain, indicating the influence of the shock waves. In the supersonic regime, the flow is more compressed, and the streamlines are closer together, particularly near the shock regions. The slight disturbances in the streamlines correspond to regions where the flow velocity undergoes rapid changes, a characteristic of supersonic flows as they approach and pass through shocks. The streamline patterns confirm that the numerical method successfully captured the supersonic flow features, including the formation of shock waves and their effects on the potential field $\varphi(x, y)$. The ability of the SOR method to handle these complex features demonstrates its robustness and applicability to supersonic flow simulations.

5.3 Effectiveness of the SOR Method

The results of this study highlight the efficacy of the SOR method in solving the SDE across different flow regimes. The study's findings are consistent with existing literature, further validating the SOR method's application in CFD for both subsonic and supersonic flows. The rapid convergence observed in the subsonic flow simulations aligns with the general understanding that elliptic equations, such as those governing subsonic flows, are well-suited for iterative methods like SOR. This observation is supported by previous studies, which have shown that the SOR method can significantly reduce the number of iterations required for convergence when compared to standard iterative methods like Gauss-Seidel. In the supersonic flow simulations, the SOR method also demonstrated strong performance, accurately capturing the shock waves and other complex flow features. The convergence behavior in the supersonic regime, where the residual error rapidly approached a near-zero value, indicates the method's robustness in handling hyperbolic equations. This finding is consistent with research by Kryeziu and Johnson (2013), who emphasized the importance of carefully selecting numerical schemes and relaxation factors when applying iterative methods to hyperbolic PDEs, particularly in the presence of shocks.

5.4 Implications for Aerodynamic Design

The ability of the SOR method to accurately simulate both subsonic and supersonic flows has significant implications for aerodynamic design. The results of this study demonstrate that the method can provide reliable insights into the behavior of airflows around different geometries, aiding in the design and optimization of airfoils, wings, and other aerodynamic structures. This capability is particularly relevant in the context of modern aerospace engineering, where the demand for high-performance, fuel-efficient designs continues to drive innovation. Moreover, the study's approach to handling boundary conditions and capturing shocks suggests that the SOR method could be integrated into more complex CFD frameworks, potentially enhancing the predictive capabilities of aerodynamic simulations. The continued development of such numerical methods will be crucial in meeting the challenges of future aerospace projects, where accurate, efficient simulations are essential for success.

5.5 Limitations and Future Work

While the SOR method has proven effective in this study, it is important to acknowledge its limitations. The method's performance is highly dependent on the choice of the relaxation factor (ω), and finding the optimal value can be challenging, particularly in complex flow regimes. Additionally, while the method performed well in the cases studied, its applicability to more complex three-dimensional flows or flows with more intricate boundary conditions remains to be fully explored. Future research could focus on extending the SOR method to three-dimensional flows, where the added complexity may require further refinement of the numerical techniques used. Additionally, integrating adaptive mesh refinement and more sophisticated shock-capturing methods could enhance the method's accuracy and efficiency, making it even more suitable for a broader range of aerodynamic applications.

6. Conclusion

This study has effectively applied the Successive Over-Relaxation (SOR) method to solve the Small Disturbance Equation (SDE) for both subsonic and supersonic flow conditions. The numerical approach adopted here, utilizing the finite difference method in conjunction with SOR, demonstrated substantial accuracy and efficiency in modeling the aerodynamic behaviors pertinent to different flight regimes. Specifically, the SOR method facilitated rapid convergence and reliable prediction of flow patterns, which are crucial for aeroelastic analyses and aerodynamic force calculations. Results from the simulations confirmed the method's capability to handle the distinct challenges posed by each flow regime, from the smoother, more predictable subsonic flows to the complex, shock-laden supersonic flows. Several avenues for further research present themselves. One promising area involves enhancing the SOR method's shock-capturing capabilities and exploring adaptive grid refinement techniques. These advancements could significantly improve the accuracy of simulations, especially in supersonic conditions where shocks and complex flow features predominate. Another potential direction is the extension of this methodology to three-dimensional flow models, which would align more closely with real-world aerodynamic challenges and provide deeper insights into the behavior of airflows around complex geometries. Additionally, investigating the integration of more complex boundary conditions and refining the approach to better handle intricate flow dynamics could further broaden the applicability and effectiveness of the SOR method in computational fluid dynamics, ultimately contributing to more sophisticated and efficient aerodynamic design tools.

References

[1] Allahviranloo T. Successive over relaxation iterative method for fuzzy system of linear equations. Applied Mathematics and Computation, 2005, 162(1): 189-196.

[2] Kryeziu O., Johnson E. Subsonic to supersonic nozzle flows. SIAM Journal on Applied Mathematics, 2013, 73: 175-194. https://doi.org/10.1137/110845434

[3] Li X. Research on 2D SDE method of fluid field effect above airfoil at transonic speed. Highlights in Science, Engineering and Technology, 2023, 76: 254-260.

[4] Qian X. The impact of inlet Mach number and domain shape

on subsonic nozzle flow via small disturbance equation. Journal of Physics: Conference Series, 2023, 2441(1): 012002. https://doi.org/10.1088/1742-6596/ 2441/1/012002
[5] Zhu Xianxun, Huang Yao, Wang Xiangyang, Wang Rui. Emotion recognition based on brain-like multimodal hierarchical perception. Multimedia Tools and Applications, 2024, 83(18): 56039-56057.
[6] Rom J. High angle of attack aerodynamics: subsonic, transonic, and supersonic flows. Springer Science & Business Media, 2012.
[7] Rozov V., Winter M., Breitsamter C. Antisymmetric boundary condition for small disturbance CFD. Journal of Fluids and Structures, 2019, 85: 229-248.
[8] Shan S., Chen W., Chen Z. A survey of application and development of small disturbance equation method in the aviation area. Journal of Physics: Conference Series, 2021, 2012(1): 012025. https://doi.org/10.1088/ 1742-6596/2012/1/012025
[9] Cheng Fanrui, Yang Xu, Zhou Wenbo. Nozzle flow simulation by small disturbance approximation and Euler method. Journal of Physics: Conference Series, 2021, 2012(1): 012025.
[10] Li Zhaorui, Jaberi Farhad A. A high-order finite difference method for numerical simulations of supersonic turbulent flows. International Journal for Numerical Methods in Fluids, 2012, 68(6): 740-766.