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Mean Value Theorem and the Inverse Problem

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Abstract:

The mean value theorem of differentiation is the core theorem of differential calculus. It's an important tool for studying functions, and a bridge between functions and derivatives. This paper briefly reviews the mean value theorem for differentiation. The Lagrange mean value theorem is the core, Rolle theorem is a special case, and the Cauchy theorem is an extension. It's the foundation of calculus theory and crucial for function research. The paper introduces the specific contents, proof methods, and applicable examples of these theorems. This paper also discusses their inverse problems with proofs. The mean value theorem has a profound impact on academic development, laying the foundation, promoting analysis, and expanding the research field. The theorem can be applied not only in functional analysis but also in biomathematics. It also nurtures scientific thinking, provides problem-solving ideas, and stimulates innovation. Finally, it's hoped that its application fields will expand, not just in math but also in other disciplines.

Keywords: Mean value theorem; differentiation; functions

1. Introduction

The mean value theorem (MVT) for differentiation is the general name of a series of MVT and is a powerful tool for studying functions. The most important content is the Lagrange theorem. It can be said that other MVTs are special cases or generalizations of the Lagrange MVT. The MVT for differentiation reflects the relationship between the locality of the derivative and the integrity of the function, and it has very extensive applications.

In 2002, The asymptotic property of the midpoint in the MVT was obtained when the length of the interval tends to zero. A new result with generality was obtained by Yang and Jia [1]. In 2005, Liu and Zhang had a further discussion regarding the asymptotic nature of the midpoint. Then several new asymptotic estimators under weaker conditions were obtained [2]. In 2007, Zhao and Wang made improvements on the first integral MVT and the generalized first integral MVT. By listing several typical questions, the improved theorems were applied to handle these problems succinctly and clearly [3]. In 2008, Zhang adopted the method of introducing parameters and gave the generalized forms of the first integral MVT and the second integral MVT respectively [4]. In the same year, Liu and Yang mainly studied the monotonicity, continuity and differentiability of the midpoint ζ in the Lagrange MVT [5]. In 2019, Zhang and his team studied the asymptotic behavior of the midpoint in the higher-order Cauchy MVT when $x \to +\infty$. Under certain conditions, two asymptotic estimators of the midpoint in the higher-order Cauchy MVT when $x \to +\infty$ are established. The results of this paper enrich the relevant results of the asymptotic nature of the midpoint in the MVT [6].

In this paper, three type of MVTs are explained, a common example is given in section 2.1. In section 2.2, some inverse problems are shown.

2. Mean Value Theorems

2.1 Brief Introduction of Three Types of Mean Value Theorems

2.1.1 Rolle's Theorem

Differential calculus is well used in the problems not only in the region of Mathematics and other sciences but also in some of the cultural subjects. To talk about calculus, a famous mathematics should be mentioned whose name is Michel Rolle (1652-1719). He claimed an important theorem about the differential calculus which called Rolle's Theorem in the year of 1691.

Next, Rolle's Theorem and its proof will be given.

Theorem 1: If function $H(x)$ satisfies these conditions below:

a. $H(x)$ is a continuous function between the closed interval α and β;

b. $H(x)$ is a differentiable function between the open interval α and β;

c. $H(\alpha)=H(\beta)$.

Then, there at least exists one point ζ in the set of (α, β) leading to $F'(\zeta)=0$ [7].

Proof: As $F(x)$ is continuous between [α, β], there are a maximum and a minimum between [α, β] which are equal to M and m respectively.

Now it is split into two conditions:

a. If $M=m$, then $f(x)$ is a constant function with $H(x)=M$ between the set of $[\alpha,\ \beta]$. Therefore, for all $\zeta\in(\alpha,\ \beta), H'(\zeta)=0$.

b. If M>m, then at least one value is unequal to the $H(\alpha)$ or $H(\beta)$. Suppose that $M\neq H(\alpha)$, then there exists at least one point such that $H(\zeta)=M$. As $f(x)$ is differentiable between $[\alpha,\ \beta]$, so according to the Fermat Theorem, $H'(\zeta)=0$.

The use of Rolle' Theorem will be shown below:

Example 1: Given $f(x)$ is continuous between $[\alpha,\ \beta]$, and is differentiable between $(\alpha,\ \beta)$. $f(0)=e$, $f(1)=1$. Prove that $\exists\zeta\in(\alpha,\ \beta) which makes f(\zeta)+f'(\zeta)=0$.

Proof: Let $F(x)=e^x f(x)$.

$F(0)=e^0 f(0)=e, F(1)=e^1 f(1)=e$. As $F(x)$ is continuous between $[0,1]$, differentiable between $(0,1)$, and $F(0)=F(1)=e$. So, according to the Rolle's Theorem, there exists a $\zeta\in(a,b)$, which makes $F'(\zeta)=0$. Then $e^{\zeta} f(\zeta)+e^{\zeta} f'(\zeta)=0$. So, when $\zeta\in(0,1), e^{\zeta}>0$. Then $f(\zeta)+f'(\zeta)=0$.

This is a typical example of using Rolle's Theorem, which is making the original function into two equal functions $F(0)$ and $F(1)$. This makes sure the function is continuous and differentiable between the intervals needed. Then using $(u+v)'=uv'+u'v$, the new equation $e^{\zeta} f(\zeta)+e^{\zeta} f'(\zeta)=0$ will be obtained. Check $e^{\zeta}\neq 0$, and divide both two side of the equation by e^{ζ}. Then the needing equation is obtained.

2.1.2 Lagrange's Mean Value Theorem

Due to the limitations of Rolle's MVT, it is not as widely used as Lagrange's MVT. Next, Lagrange's MVT and its proof will be given.

Theorem 2: In Lagrange's MVT, if $h(x)$ satisfies bellowing conditions:

a. $h(x)$ is a continuous function between $[\alpha,\ \beta]$

b. $h(x)$ is a differentiable function between $(\alpha,\ \beta)$

Then there must at least have a point ζ between (α, β) so that:

$$h'(\zeta)=\frac{f(\beta)-f(\alpha)}{\beta-\alpha} \tag{1}$$

The function $y=h(x)$ is a continuous smooth curve between $[\alpha,\beta]$, so there at least has one point that has same gradient with cord AB in the arc AB. If $h(\alpha)=h(\beta)$, the gradient of cord AB is 0. So, $f'(\zeta)=0$, which is same as that in the Rolle's theorem.

Proof: A common way to prove it is by creating an auxiliary function [8]:

$$\phi(x)=h(x)-h(\beta)-\frac{h(\beta)-h(\alpha)}{\beta-\alpha}(x-\alpha), x\in[\alpha,\beta] \tag{2}$$

As function $h(x)$ is continuous between $[\alpha,\beta]$, and differentiable between (α,β), it has:

$$\phi(\beta)=\phi(\alpha)=0 \tag{3}$$

Then according to the Rolle's theorem, there at least has a point $\zeta ?(\alpha,\beta)$, which makes that $\phi(\zeta)=0$. Taking the derivative of the expression of $\phi(\zeta)$ and ϕ ¡®$(\zeta)=0$, then we have:

$$h'(\zeta)=\frac{h(\alpha)-h(\beta)}{\alpha-\beta} \tag{4}$$

The use of Lagrange's theorem will be shown below:

Example 2: Let $f(x)$ be continuous between $[0,1]$, and differentiable between $(1,0)$. Given that $3\int_{\frac{2}{3}}^{1} f(x)dx=f(0)$. Prove there exists $\zeta\in(0,1)$ such that $f(\zeta)=0$.

Proof: According to the MVT, there exists $c\in[\frac{2}{3},1]$, which

$$\int_{\frac{2}{3}}^{1} f(x)dx=f(c)(1-\frac{2}{3}) \tag{5}$$

Therefore

$$f(c)=3\int_{\frac{2}{3}}^{1} f(x)dx=f(0) \tag{6}$$

Using Rolle's theorem for $f(x)$ in $[0,c]$, there exists $\zeta\in(0,c)\in(0,1)$ making $f'(x)=0$.

2.1.3 Cauchy's Mean Value Theorem

After two typical types of MVT, Cauchy's MVT will be given.

Theorem 3: ([9]) If function $f(x)$ and $g(x)$ satisfy belowing condition:

a. Continuous between $[a,b]$;

b. Differentiable between (a,b)

c. $\forall x \in (a,b)$, with $g'(x) \neq 0$

Then there at least exists a point c between (a,b) that make $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

After the brief introduction of the theorem, text below is the reason why this theorem is true.

Proof: Firstly, prove $g(b)-g(a) \neq 0$:

By contradiction, assume that $g(b)-g(a)=0$, so, $g(b)=g(a)$. According to the Rolle's theorem, there must have a point c between (a,b), that makes $g'(c)=0$. Here appears a contradiction.

So, the origin statement is false since $g(b)-g(a) \neq 0$. $g(a) equalsto A$, $g(b) equalsto B$, and $A \neq B$

As $\forall x \in (a,b)$, $g'(x) \neq 0$. So, we can know that $g'(x)$ will always be bigger than 0 or always smaller than 0. In fact, if there exist $x_1, x_2 \in (a,b)$, let's assume that $x_1 < x_2$, then $g'(x_1) < 0$, $g'(x_2) > 0$. according to the Darboux theorem, so there must exists $x_0 belongingto (a,b)$ which $g'(x_0)=0$. It is contradicted with known condition. Thus, there always holds $g'(x) > 0$ or $g'(x) < 0$ between (a,b).

Let's assume $g'(x) > 0$, then $g(x)$ is a strict single tone increment function. It can be seen from the existence theorem of inverse function, the continuity theorem of inverse function and the derivation rule of inverse function that on $[A,B]$ there exists the inverse function $y = g(x)$. It is written as $x = g^{-1}(y)$, which is single tone increment. In addition, $x = g^{-1}(y)$ is differentiable between (A,B), and the differential function is $[g^{-1}(y)]^{-1} = \frac{1}{g^{-1}(x)}$.

Let

$$F(y) = f(g^{-1}(y)), y \in [A,B] \quad (7)$$

Then

$$F(B) = f(g^{-1}(B)) = f(b), F(A) = f(g^{-1}(A)) = f(a) \quad (8)$$

Since $F(y)$ is formed by the complex combination of $f(x)$ and $x = g^{-1}(y)$, it is known that $F(y)$ continues in the interval $[A,B]$, differentiates in (A,B), satisfies the Lagrange's median theorem, and

$$F'(y) = f'(x)[g^{-1}(y)]' = \frac{f'(x)}{g'(x)} \quad (9)$$

$F'(y)$ should be reasoned with the Lagrange median on $[A,B]$, then it can be known that there is at least one point η in the open interval (A,B), so that

$$F'(\eta) = \frac{F(B)-F(A)}{B-A}(A < \eta < B) \quad (10)$$

Let $C = g^{-1}(\eta)$, and pay attention to $x = g^{-1}(y)$. By $A < \eta < B$, it leads to $A < C < B$. Substituting (9), then we can get

$$F'(\eta) = F'(y)\ |_{y=\eta} = \frac{f'(x)}{g'(x)}\ |_{x=c} = \frac{f'(c)}{g'(c)} a < c < b \quad (11)$$

Substituting (8) and (11) into (10), then $\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$. The proposition is proved.

Next, the application of Cauchy's theorem will be given.

Example 3: Assume that f is a continuous function between $[\alpha,\beta] with \alpha > 0$, differential between (α,β). Then, there exists $\zeta \in (\alpha,\beta)$ that holds $f(\beta)-f(\alpha) = \zeta f'(\zeta) ln\frac{\beta}{\alpha}$.

Proof: The equation to be proved can be changed into: $\frac{f(\beta)-f(\alpha)}{ln\beta - ln\alpha} = \frac{f'(\zeta)}{\text{\raisebox1}ex\$1\$/\text{\raisebox}-1ex\$\zeta\$}$.

Let $g(x) = lnx$, it is clear that the function $f(x)$ and $g(x)$ are satisfying the condition of Cauchy's MVT. So there exists $\zeta \in (\alpha,\beta)$ that makes $\frac{f(\beta)-f(\alpha)}{ln\beta - ln\alpha} = \frac{f'(\zeta)}{\text{\raisebox1}ex\$1\$/\text{\raisebox}-1ex\$\zeta\$}$.

Which is equivalent to $f(\beta)-f(\alpha) = \zeta f'(\zeta) ln\frac{\beta}{\alpha}$.

2.2 The Inverse Problem of the Mean Value Theorem

Lemma: If function $h(x)$ is continuous between $(\zeta-\delta, \zeta+\delta)$ and can get a strict extremum at ζ. Then for any $0 < \eta < \delta$, there exist $x_1, x_2 \in (\zeta-\eta, \zeta+\eta)$. It holds that $h(x_1) = h(x_2)$.

2.2.1 The Inverse Problem of Lagrange's Mean Value Theorem

Theorem 4: If function $h(x)$ satisfies:

a. Continuous between $[a,b]$;

b. Differential between (a,b);

c. $h''(\zeta)$ is existed that $\zeta belongingto(a,b)$, and $h''(\zeta)\neq 0$;

Then there exist $x_1,x_2\in(a,b)x_1<\zeta<x_2$, that satisfy:

$$\frac{h(x_2)-h(x_1)}{x_2-x_1}=h'(\zeta) \tag{12}$$

Proof: Let suppose $h''(\zeta)>0$. Then it is divided into two conditions.

(I) If $h'(\zeta)=0$, as $h''(\zeta)>0$, ζ is the strict minimum point. Then there exists a constant $\delta>0,[\zeta-\delta,\zeta+\delta]\in(a,b)$. Then there holds $h(\zeta)<h(x)$ with $\zeta-\delta\leqslant x\leqslant\zeta+\delta, x\neq\zeta$.

If $h(\zeta-\delta)=h(\zeta+\delta)$, then let $x_1=\zeta-\delta, x_2=\zeta+\delta$. $h(\zeta-\delta)\neq h(\zeta+\delta)$. Suppose that $h(\zeta-\delta)<h(\zeta+\delta)$, $x_1=\zeta-\delta$, then $h(\zeta)<h(x_1)=h(\zeta-\delta)<h(\zeta+\delta)$. According to intermediate value theorem for continuous function, there exists $x_2\in(\zeta,\zeta+\delta)$ that makes $h(x_2)=h(x_1)$. Then $\frac{h(x_2)-h(x_1)}{x_2-x_1}=0=h'(\zeta)$.

(II) If $f'(\zeta)\neq 0$, let $g(x) equalstof(x)-f'(\zeta)x$. Then there has $g'(\zeta)=0$. According to (I), there exist two points $x_1,x_2\in(a,b)$. It holds that $\frac{g(x_2)-g(x_1)}{x_2-x_1}=0$, which $\frac{h(x_2)-h(x_1)}{x_2-x_1}=h'(\zeta)$.

2.2.2 The Inverse Problem of Cauchy's Mean Value Theorem

Theorem 5: If $f(x), g(x)$ satisfy:

a. Continuous between $[a,b]$;

b. It is differentiable in a second order with symmetry between (a,b);

c. For any $\zeta\in(a,b)$, there have $f''(x)g'(x)-f'(x)g''(x)\neq 0 and g'(x)\neq 0$.

Then for any $\zeta\in(a,b)$, there exist $x_1,x_2\in(a,b), x_1<\zeta<x_2$ such that

$$\frac{f(x_2)-f(x_1)}{g(x_2)-g(x_1)}=\frac{f'(\zeta)}{g'(\zeta)} \tag{13}$$

Proof: Let $h(x)=\frac{f'(x)}{g'(x)}$, then according to the known condition, $h(x)$ is differentiable with symmetry between (a,b) and $h'(x)\neq 0$. Let $h'(x)>0$ and $g'(x)\geqslant 0$, then $h(x)$ is strictly monotonically increasing between (a,b)[10].

Construct a function $F(x)=g'(\zeta)f(x)-f'(\zeta)g(x)$. According to the monotonicity of $h(x)$ between (a,b) and Cauchy MVT, for any $x_1,x_2\in(a,b)$, when $x_1<x_2$, there has:

$$f(x_2)-f(x_1)=\frac{f'(\zeta)}{g'(\zeta)}(g(x_2)-g(x_1))>\frac{f'(x_1)}{g'(x_1)}(g(x_2)-g(x_1)) \tag{14}$$

This equals to $g'(x1)(f(x2)-f(x1))>f'(x1)(g(x2)-g(x1))$.

When $x>\zeta, F(x)-F(\zeta)=g'(\zeta)(f(x)-f(\zeta))-f'(\zeta)(g(x)-g(\zeta))>0$.

When $x<\zeta, F(x)-F(\zeta)=g'(\zeta)(f(x)-f(\zeta))-f'(\zeta)(g(x)-g(\zeta))<0$.

Because $F'(x)=g'(\zeta)f'(\zeta)-f'(\zeta)g'(\zeta)=0$, $F'(x)$ has it maximum or minimum at ζ. There exist $x_1,x_2\in(a,b)$, $x_1<\zeta<x_2$, which leads to $F(x_1)=F(x_2)$.

This equals to

$$g'(\zeta)f(x_1)-f'(\zeta)g(x_1)=g'(\zeta)f(x_2)-f'(\zeta)g(x_2) \tag{15}$$

That is:

$$\frac{f(x_2)-f(x_1)}{g(x_2)-g(x_1)}=\frac{f'(\zeta)}{g'(\zeta)} \tag{16}$$

The proposition is proved now.

3. Conclusion

The paper first gives a brief review of the MVT for differentiation, pointing out that the Lagrange MVT is the core content of the MVT for differentiation, the Rolle theorem is its special case, and the Cauchy theorem is its generalization. The MVT for differentiation is the foundation of the calculus theory and plays an important role in the study of functions. Then, the specific contents and proof methods of the Rolle theorem, the Lagrange MVT and the Cauchy theorem will be given. The examples applicable to each theorem are introduced. The Cauchy theorem further generalizes the Lagrange MVT and is applicable to the case of two functions. The paper also discusses the inverse problems of the Cauchy MVT and the Lagrange MVT and gives the proof of the relevant theorems. The MVT is an important theorem in differential calculus, which has many profound impacts on the development of academia. It promotes the deepening of mathematical theory: laying the foundation of calculus and promoting the development of analysis. It expands the field of mathematical research not only in the field of functional analysis, but also in biomathematics. It also cultivates the way of thinking in scientific research, provides ideas for problem-solving, and stimulates innovative thinking. It is hoped that with the development of the times, the fields not only mathematics but also other subject that need the MVT can be widely spread out.

References

[1] Yang C., Jia Y. A general result regarding the mean value theorem of integration. Practice and Understanding of Mathematics, 2002, 32(4): 3.

[2] Liu X., Ai Q., Zhang S. On the asymptotical nature of the intermediate point in the mean value theorem. Journal of Bohai University (Natural Science Edition), 2005.

[3] Zhao W., Wang G. The improved first mean value theorem of integration and its applications. Journal of Xinjiang Normal University: Natural Science Edition, 2007, 26(2): 4.

[4] Zhang Q. The generalization of the mean value theorem of integration. Journal of Shangqiu Normal University, 2008, 24(6): 3.

[5] Liu L., Dai L., Yang Z. Further discussion on the properties of the intermediate point ξ in the differential mean value theorem. College Mathematics, 2007, 23(4): 163-166.

[6] Zhang X., Zhang S. The asymptotical property of the “intermediate point” in the higher-order Cauchy mean value theorem. Journal of Ludong University (Natural Science Edition), 2019.

[7] Zhao Y., Chen Z., Liu H. et al. Rohr’s mean value theorem and its teaching annotation [J]. Modernization of Education, 2021(5): 81-82.

[8] Qu J., Zhang J. Proof and application of Lagrange’s mean value theorem. Taiyuan Normal University. 2021.

[9] Jing T. Cauchy’s mean value theorem and its applications. Science and Technology Information (Academic Research), 2008, (27): 91-92.

[10] Li H. The mean value theorem for symmetric derivatives and its inverse problem. Journal of Guangzhou Open University. 2013, (1): 4.