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Deductions, Applications and Expansion of the Cauchy's Residue Theorem

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Abstract:

Cauchy's integral formula is one of the most important discovery in the development history of the complex variable integration. This article focuses on the methods of the integration in mathematics. The author aims to write about how to deduce the residue theorem from Cauchy's integral formula and how to use the Cauchy's integral theorem and residue theorem in the integral questions. The following parts of the article include using method of limits and Euler's formula to deduce roughly the Cauchy's Integral formula. In addition, the author also uses the differentiation to transfer the integral function. In part three, the author calculates three different types of examples and considers that residue theorem and Cauchy's integral formula are very useful in the contour integral with singularities. The article is in order to show that the practicability of these theorem and formula. These formulas and theorems are used in a wide range of the subjects and areas.

Keywords: Cauchy's residue theorem; Cauchy's integral theorem; Laurent expansion.

1. Introduction

When the Augustin-Louis Cauchy was in his 24, he proposed the Cauchy's integral theorem in a piece of paper which he gave to the Académie des Sciences on August 11, 1814. Then, he completed the full theorem in 1825. In the next year, the Cauchy pointed out the definition of the residue which laid the foundation of the residue theorem. However, in 1831, he sent two papers to the Académie des Sciences. The first one contained the Cauchy's integral theorem. and the second one pointed out the residue theorem.

Recent years, many mathematicians focus on the proof and the use of the residue theorem. For example, the reference [1] shows the complete proof of the residue theorem. The writer of the reference [1] said that 'the Cauchy's residue theorem is relative to its intermediate results — the argument principle and Rouché's theorem'. This shows the scalability and the importance to the complex variable functions. Reference [2] is about civil engineering which means that the residue theorem can not only use in mathematics but also use in physics. The author said that 'in order to solve the problem that the solution of the coupling coefficient integration in the multidimensional multipoint response spectroscopy is too complicated. Basing on residue theorem, someone points out a sufficient method. This method not only considers the several types of the coupling coefficient and transfers the coupling coefficient formula into analytic form, but also has the equivalent result and is more than 100 times sufficient than the integral methods. This shows the convenience of the residue theorem. In addition, in reference [3], the topic is about the topological phase transitions that are also about physics. In this reference, the author mentions several Su-Schrieffer-Heeger models to illustrate the applications of the residue theorem in topological phase transitions. All these articles show the practicability of the residue theorem.

The author based on the method in the reference [4] deduce the Cauchy's integral theorem. In order to solve the special type of the integral more convenient and sufficient, the author writes three types of integrals which can solve them in a more convenient way than the normal methods using the following formulas. In example 1 from reference [5], the author uses the Euler formula and the residue theorem to calculate the integral that contain $sin\theta$ in a convinient way. Personally, the author considers that one of the best methods to calculate the integral with trigonometric function such as $sin\theta$ and $cos\theta$ is using the Euler's formula and calculate the real and image part seperately. Then, taking the real part or the image part is decided by the trigonometric function in the integral. In example 2 from reference [6], the author separates the integral into two situation and discusses both situations. In example 3 from reference [7], if the author uses the residue theorem, the author needs to discuss 4 different situations. The last part is conclusion of the whole articles.

2. Theorem and Method in Residue Theorem

2.1 Background of Residue Theorem

Firstly, the Cauchy's residue theorem in residue is proved by using Cauchy's integral formula [3]

$$f\left(z_0\right)=\frac{1}{2\pi i}\oint_C\frac{f\left(z\right)}{z-z_0}dz. \quad (1)$$

The region surrounded by C is the simply connected region of $f\left(a\right)$ and a is a point in the region.

Proof. The integral of the connected region C can transfer to the integral of the closed region C_r surrounding the point which is in the region C, As shown in the Fig. 1. The author shows that $\oint_c\frac{f\left(z\right)}{z-z_0}dz=\oint_{c_r}\frac{f\left(z\right)}{z-z_0}dz$ where $z\in C_r$. Let $z=z_0+cos\theta+isin\theta=z_0+R_2e^{i\theta}$, by using the Euler's formula, it is found that

$$\oint_{c_r}\frac{f(z)}{z-z_0}dz=\oint_0^{2\pi}\frac{f\left(z_0+R_2e^{i\theta}\right)}{z_0+R_2e^{i\theta}-z_0}dz=\oint_0^{2\pi}\frac{f\left(z_0+R_2e^{i\theta}\right)}{R_2e^{i\theta}}iR_2e^{i\theta}d\theta \quad (2)$$

Thus,

$$\oint_{c_r}\frac{f(z)}{z-z_0}dz=i\oint_0^{2\pi}f\left(z_0+R_2e^{i\theta}\right)d\theta \quad (3)$$

The size of the circle does not have any effect to the answer. As a result, the author can let the r approach zero. Therefore,

$$\oint_c\frac{f(z)}{z-a}dz=i\lim_{r\to0}\oint_0^{2\pi}f\left(z_0+re^{i\theta}\right)d\theta=i\oint_0^{2\pi}f\left(z_0\right)d\theta=f\left(z_0\right)2\pi i \quad (4)$$

So, $f\left(z_0\right)=\frac{1}{2\pi i}\oint_C\frac{f\left(z\right)}{z-z_0}dz$.

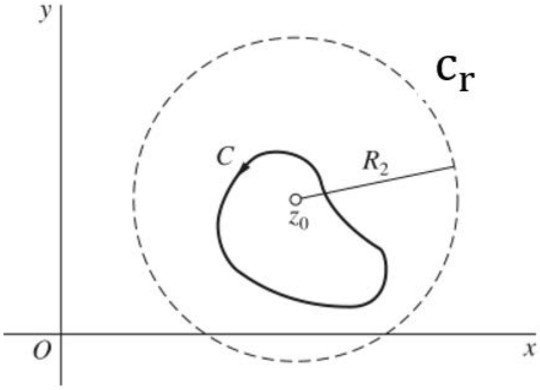

Fig. 1 Illustration of the contour in the complex plane.

2.2 Proof of Laurent Expansion

The function $f\left(z\right)$ is resolved in a ring region centered on z_0. z is a point in this area. The outer circle is C_2 and the inner circle is C_1. As a result, the author can reconstruct the integral loop. The author makes a cut line. A and A' are same point. The integral of A' to B' equals to the integral of $\ddot{C}\left(BtoA\right)$. Thus, the author can turn the integral of loop to the integral of A to A' to B' to B to A. Finally, the integration is the integral of the counterclockwise direction along C_2 plus the integral of the clockwise direction along C_1. This idea is demonstrated in Fig. 2.

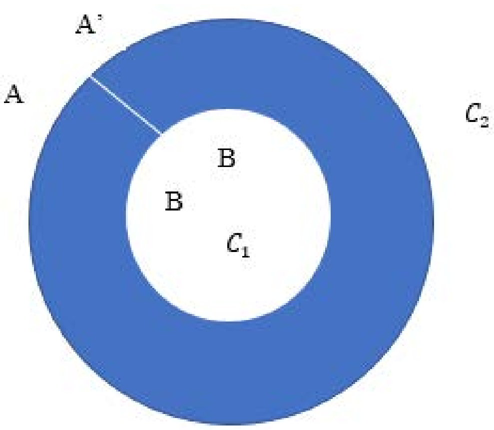

Fig. 2. An integral loop with a cut line between A and B.

Because of the continuity of the integration, the author uses the theorem 1 to get

$$f(\mathrm{z})=\frac{1}{2\pi\mathrm{i}}\oint_{\mathrm{L}}\frac{f(\varepsilon)}{\varepsilon-z}\mathrm{d}\varepsilon=\frac{1}{2\pi\mathrm{i}}\oint_{\mathrm{C}_2}\frac{f(\varepsilon)}{\varepsilon-z}\mathrm{d}\varepsilon-\frac{1}{2\pi\mathrm{i}}\oint_{\mathrm{C}_1}\frac{f(\varepsilon)}{\varepsilon-z}\mathrm{d}\varepsilon \tag{5}$$

where

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(\epsilon)}{\epsilon-z}d\epsilon=\frac{1}{2\pi i}\oint_{C_2}\frac{f(\epsilon)}{(\epsilon-z_0)-(z-z_0)}d\epsilon=\frac{1}{2\pi i}\oint_{C_2}\frac{f(\epsilon)}{\epsilon-z_0}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\epsilon-z_0)^n}d\epsilon. \tag{6}$$

Since $-\frac{1}{2\pi i}\oint_{C_1}\frac{f(\epsilon)}{\epsilon-z}d\epsilon=\frac{1}{2\pi i}\oint_{C_2}\frac{f(\epsilon)}{z-z_0}\frac{1}{\left(1-\frac{\epsilon-z_0}{z-z_0}\right)}d\epsilon$ is satisfied and noticing the fact that the relation $-\frac{1}{2\pi i}\oint_{C_1}\frac{f(\epsilon)}{\epsilon-z}d\epsilon=\frac{1}{2\pi i}\oint_{C_2}\frac{f(\epsilon)}{(z-z_0)}\sum_{n=0}^{\infty}\frac{(\epsilon-z_0)^n}{(z-z_0)^n}d\epsilon$ holds, thus

$$f(z)=\sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\oint_{C_2}\frac{f(\epsilon)}{(\epsilon-z_0)^{n+1}}d\epsilon\right](z-z_0)^n+\sum_{n=-\infty}^{-1}\left[\frac{1}{2\pi i}\oint_{C_2}\frac{f(\epsilon)}{(\epsilon-z_0)^{n+1}}d\epsilon\right](z-z_0)^n. \tag{7}$$

This indicates that [5]

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n \tag{8}$$

where $a_n=\frac{1}{2\pi i}\oint_{C_2}\frac{f(\epsilon)}{(\epsilon-z_0)^{n+1}}d\epsilon$.

2.3 Definition of the Residue

$\mathrm{Res}\left[f(z_0)\right]$ is called the residue of the point z_0. Residue on the point z_0 is a_{-1} by $f(z)$ using the Laurent expansion in the centerless field of z_0.

Proof. Supposed that $f(z)$ has only one isolated singularity z_0 in the area enclosed by the integral area, and C_r is a circle with the center z_0 and the radius R. Using the Cauchy's integral theorem, the author transforms the integral of the large area C to the integral of the small circle C_R, namely, $I=\oint_c f(z)dz=\oint_{C_R} f(z)dz$. Using the Laurent expansion shown in Eq. (8), and in the circle C_R, $z=z_0+Re^{i\theta}$ and $dz=iRe^{i\theta}d\theta$. Thus,

$$I=i\sum_{n=-\infty}^{\infty}a_nR^{n+1}\int_0^{2\pi}e^{i(n+1)\theta}d\theta. \tag{9}$$

When $n=-1$, the function $\int_0^{2\pi}e^{i(n+1)\theta}d\theta$ equals to the 1. If $n\neq-1$, the function $\int_0^{2\pi}e^{i(n+1)\theta}d\theta$ equals to the integral of the $cos\left[(n+1)\theta\right]+isin\left[(n+1)\theta\right]$ in the range of 0 to 2π which equals to 0. Thus,

$$\int_0^{2\pi}e^{i(n+1)\theta}d\theta=2\pi\delta_{n,-1} \tag{10}$$

Since $I=2\pi i\sum_{n=-\infty}^{\infty}a_nR^{n+1}\delta_{n,-1}=2\pi ia_{-1}$, it is found that

$$I=\oint_c f(z)dz=2\pi iResf(z_0). \tag{11}$$

3. Example of the Residue Theorem

3.1 Calculate the $\int_0^{+\infty}\frac{x\mathbf{sin}x}{x^2+a^2}dx$

Because of the Euler's formula, one finds that [2]

$$\int_{-\infty}^{\infty}f(x)e^{ix}dx=\int_{-\infty}^{\infty}f(x)cosxdx+\int_{-\infty}^{\infty}f(x)isinxdx \tag{12}$$

In this integral equation, $f(x)$ is $\frac{x}{x^2+a^2}$. If $x^2+a^2=0$, $f(x)$ has a one over two poles. Thus, $x=ai$ is the pole. Then, the author uses the theorem shown in Eq. (11). Thus, $\int_{-\infty}^{\infty}f(x)e^{ix}dx=2\pi i\bullet\frac{1}{2}\bullet e^{i\times ai}=\pi ie^{-i}$.

This answer only has imaginary part, so the author can get the answer of $\int_{-\infty}^{\infty}f(x)isinxdx$ by separating the imaginary part and the real part. From the equation, the author knows that this function is an even function which mains that the function is symmetry of the y-axis, and thus $\int_{-0}^{\infty}f(x)isinxdx=\int_{-\infty}^{0}f(x)isinxdx$. Finally, the author can get the answer of the $\int_0^{+\infty}\frac{xsinx}{x^2+a^2}dx$ which equals to the half of the $\int_{-\infty}^{\infty}f(x)sinx$, and the answer is $\frac{\pi}{2e}$.

3.2 Calculate

$I=\oint_C \frac{dz}{(z-3)(z+i)^{10}(z-z_0)}\ C:|z|=2,|z_0|\neq 2, z_0\neq 3$

Firstly, the author lets $f(z)=\frac{1}{(z-3)(z+i)^{10}}$ and because the point z_0 can be many places in the are C, the area which surrounded by C equals to R and the area which not surrounded by C equal to R^-. This equation has many poles. If $z=3$, that is a simple pole of the equation. If $z=\infty$, that is a moving singularity and zero pole of the equation. As a result, the author separates this into two situations: $z\in R$ and $z\in R^-$. When $z\in R$, $I=2\pi i[Res(z=3)+Res(z=\infty)]$.

The $Res(z=3)$ and $Res(z=\infty)$ separately equal to the $\frac{1}{(z-3)(3+i)^{10}}$ and 0. Thus,

$$I=\frac{2\pi i}{(z-3)(3+i)^{10}} \tag{13}$$

When $z\in R^-$,

$$I=2\pi i[Res(z=3)+Res(z=\infty)-f(z_0)] \tag{14}$$

As same as before, the integral is found to be

$$I=2\pi i\left[\frac{1}{(z-3)(3+i)^{10}}+\frac{1}{(z_0-3)(z_0+i)^{10}}\right]. \tag{15}$$

3.3 Calculate $\oint_C \frac{\mathbf{e^z}}{\left(\mathbf{z}(1-\mathbf{z})^3\right)}dz$, Where **C** is a Closed Smooth Curve Except at 0 and 1

The author discusses four situations of the curve include the point 0 and 1 or not to solve this question.

Firstly, when the curve does not include the point 0 and 1. The function $f(z)=\frac{e^z}{\left(z(1-z)^3\right)}$ belongs to the area C. There is no singularity in the area. Thus, according to the Cauchy's theorem,

$$\oint_C \frac{e^z}{\left(z(1-z)^3\right)}dz=0 \tag{16}$$

Secondly, when the curve includes the point 1 but not include point 0. Thus, the function $f(z)=\frac{e^z}{z}$ situates in the area C. The function $f(z)=\frac{e^z}{\left(z(1-z)^3\right)}$ has the only pole which is the point $z_0=1$. And, point $z_0=1$ is a third ordered singularity. Thus, $\oint_C \frac{e^z}{\left(z(1-z)^3\right)}dz=\oint_C \frac{e^z\bullet z^{-1}}{(1-z)^3}dz$. By using the Cauchy's theorem, the author can transform the equation into another form which is $2\pi i f(z)=\oint_C \frac{f(\epsilon)}{\epsilon-z}dz$. Because $f(\epsilon)$ and ϵ are both constant numbers, the only variable is z on the right-hand side. So, the author differentiates the both side of the formula [7]. Thus,

$$2\pi i f^n(z)=n!\oint_C \frac{f(\epsilon)}{(\epsilon-z)^{n+1}}dz \tag{17}$$

As a result of it, the n-th derivative of the $f(z)$ is $f^n(z)=\frac{n!}{2\pi i}\oint_C \frac{f(\epsilon)}{(\epsilon-z)^{n+1}}dz$. In the function $\oint_C \frac{e^z\bullet z^{-1}}{(1-z)^3}dz$, the author lets the $f(z)$ equals to the $e^z\bullet z^{-1}$. Thus, when $z=1$,

$$\oint_C \frac{e^z\bullet z^{-1}}{(1-z)^3}dz=-\oint_C \frac{e^z\bullet z^{-1}}{(z-1)^3}dz=\frac{-2\pi i}{2}e^1=-e\pi i \tag{18}$$

Thirdly, when the curve includes the point 0 but not include point 1. The function $f(z)=\frac{e^z}{(1-z)^3}$ situates in the area C. The function $f(z)=\frac{e^z}{\left(z(1-z)^3\right)}$ has only singularity which is $z_0=0$. By using the Cauchy's formula, one finds that

$$\oint_C \frac{\mathrm{e}^z}{\left(\mathrm{z}(1-z)^3\right)}dz=\oint_C \frac{\mathrm{e}^z/(1-z)^3}{z}dz=2\pi i. \tag{19}$$

Lastly, when the curve does not include the point 0 and point 1. The author draws the circle C_1 and C_2 that have no intersection with the center 0 and 1. In addition, the radii is $r>0$, and C_1 and C_2 are both in the region C. By using the Cauchy's integral formula for the complex perimeter,

$$\oint_C \frac{e^z}{\left(z(1-z)^3\right)}dz=\oint_{C_1} \frac{e^z}{\left(z(1-z)^3\right)}dz+\oint_{C_2} \frac{e^z}{\left(z(1-z)^3\right)}dz \tag{20}$$

By using the result of the second answer and the third answer, the formula $\oint_{C_1} \frac{e^z}{\left(z(1-z)^3\right)}dz$ equals to $2\pi i$ and the formula $\oint_{c_2} \frac{e^z}{\left(z(1-z)^3\right)} dz$ equals to $\ddot{}Ce\pi i$. Thus,

$$\oint_C \frac{e^z}{\left(z(1-z)^3\right)} dz = 2\pi i - Ce\pi i = (2-e)\pi i. \quad (21)$$

4. Conclusion

On the one hand, in the part two, the author lists three formula and theorems and prove them sketchy by using the mathematic methods. With these formula and theorem, the author deduces several expended formulas which is propose by the other mathematicians. Without a doubt, there will be more and further formula and discovery based on these theorems. In part 3, the author uses the Cauchy's integral theorem, Laurent expansion and residue theorem to solve the three integral questions. As a result of the example one, the integral in the form of the $\oint_C f(x) sinxdx$ or the $\oint_C f(x) cosxdx$ can transfer into a simple form and turn the hard question into easier one with the Euler's formula. In addition, with the developing world and mathematics, modern mathematicians may propose more related theorem and methods in order to solving the problem. This article is not so clear in several places because it includes the complicated knowledge. In the future, the author will continuously work on the area of the complex variable and the integral function.

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