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Cauchy's Residue Theorem and Its Application

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Abstract:

One of the key theorems in complex analysis, Cauchy's Residue Theorem, which refers to the integral value of an analytic function along any simple closed contour surrounding an isolated singularity in a certain ring domain divided by 2πi, will greatly simplify the process of computing integrals on contour surrounding singularities. In the field of basic mathematics, the Cauchy' Residue Theorem plays a key role in the integral calculation of analytic functions, non-uniform complex functions and the argument principle, which establishes a connection between a curve's winding number and the quantity of zeros and poles inside the curve. Furthermore, Residue Theorem can combine with various subjects, including electromagnetism. This paper gives an overview of the Cauchy's Residue Theorem and its application. Firstly, it talks about the definition and proof of the Cauchy' Residue Theorem, along with the definition of residue. Then two examples of using Residue Theorem to integrate functions will be given. Finally, this paper involves application of Residue Theorem in trigonometric sum identities and a remark on the method.

Keywords: Cauchy's residue theorem; application; trigonometric sum identities.

1. Introduction

The Cauchy's Residue theorem tells us that integrals can be evaluated by simply finding out the coefficients of the function in Laurent series, under certain broad circumstances [1]. Researchers have discovered its importance in basic theory such as Cauchy's integral formula and Watson transform [2]. The theorem also plays an important role in different fields of science. For example, in physics, it can be applied to improper integrals encountered in damped vibration, Fresnel diffraction and thermal conduction [3]. Although there were lots of research on first-order singularities, study on high-order singularities is rather rare. In 2016, Sheng summarized the methods of using residue theorem to calculate some generalized integrals and special definite integrals [4]. In 2020, Qiu used the generalized residue theorem to solve Riemann Hilbert problems with multiple high-order poles [5]. In 2022, Meng and Guan explained how to derive the expression of second-order displacement analytically with the usage of the residue theorem [6]. In 2023, Song used the generalized residue theorem to solve Riemann Hilbert problems with multiple high-order poles [7].

The main objectives of this paper are showing the opinions of the author towards the proof of Residue Theorem and its applications. Section 2.1 displays the definition and the proof of this theorem, 2 examples will be given to help understanding. Section 2.2 includes a remark and an application of the theorem in trigonometric identities.

2. Cauchy's Residue Theorem

2.1 Residues and Cauchy's Residue Theorem

Definition 1: [8] There exists a positive R_2 such that f is analytic at every point z for which $0<|z-z_0|<R_2$, when z_0 is an isolated singular point of a function f, suggesting that both the function and its derivatives turn out to be zero at z_0. As a result, f(z) has a complex number representation in the Laurent series. b_1 is the coeffi c ient of 1/(z-z_0)

Theorem 1: The function

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_n}{(z-z_0)^n}+\cdots$$

$(0<|z-z_0|<R_2)$ is called the residue of f(z), Res[f(z), z_0]= b_1.

Theorem 2: [8](Cauchy's Residue Theorem) Let C be a simple closed contour, described in the positive sense. With the exception of a finite number of isolated singular points z_k (k=1,2, ..., n) inside the contour, if a function f is analytic both inside and on C, then

$$\int_C f(z)\,dz=2\pi i\sum_{k=1}^{n} Res[f(z),z_k] \quad (1)$$

Proof :

The circles C_k described in the positive sense are internal to C and are so small that no two of them have points in common. They have the center points z_k (k=1,2, $\cdots, n$), which are isolated singular points. The closed domain formed by circles C_k and the simple closed contour C together inside which f is analytic, comprises the points inside C, not including the region of interior C_k. Hence, in accordance with the adaption of the Cauchy-Goursat theorem, the integral of the closed region where f is analytic equals to zero [8],

$$\int_C f(z) f(z) dz - \sum_{k=1}^{n} \int_{C_k} f(z) dz = 0 \quad (2)$$

$$\int_C f(z) dz = \sum_{k=1}^{n} \int_{C_k} f(z) dz \quad (3)$$

Thus, the integral of f(z) along contour C equals to the sum of the integrals of f(z) along C_k. Then, f can be expanded in Laurent Series.

Around z_1:

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_1)^m + \sum_{j=1}^{\infty} b_j (z - z_1)^{-j} \quad (4)$$

$$\oint_{C_1} f(z) dz = \sum_{m=0}^{\infty} \oint a_m (z - z_1)^m dz + \oint_{C_1} \frac{b_1}{z - z_1} dz + \sum_{j=2}^{\infty} \oint_{C_1} b_j (z - z_1)^{-j} dz. \quad (5)$$

By Cauchy's Theorem, the analytic part $\sum_{m=0}^{\infty} \oint a_m (z - z_1)^m dz$ becomes zero, so the equation is;

$$\oint_{C_1} f(z) dz = \oint_{C_1} \frac{b_1}{z - z_1} dz + \sum_{j=2}^{\infty} \oint_{C_1} b_j (z - z_1)^{-j} dz \quad (6)$$

For the second part of the right-hand side, on $C_{1,}$ Let $z = z_1 + r_1 e^{i\theta}, 0 \leqslant \theta \leqslant 2\pi$.

So, $dz = i r_1 e^{i\theta} d\theta$,

$$\sum_{j=2}^{\infty} \oint_{C_1} b_j (z - z_1)^{-j} dz = \oint_0^{2\pi} b_j \left(r_1 e^{i\theta}\right)^{-j} (i r_1 e^{i\theta}) d\theta \quad (7)$$

$$= \frac{i b_j}{r_1^{j-1}} \bullet \frac{-1}{i(j-1)} \bullet [1-1]$$

$$= 0$$

So, the equation (6) becomes,

$$\oint_{C_1} f(z) dz = \oint_{C_1} \frac{b_1}{z - z_1} dz \quad (8)$$

By Cauchy Integral Formula,

$$\oint_{C_1} f(z) dz = \oint_{C_1} \frac{b_1}{z - z_1} dz = 2\pi i \bullet b_1 \quad (9)$$

Thus, if f(z) is expanded around z_k,

$$\oint_{C_k} f(z) dz = \oint_{C_k} \frac{b_k}{z - z_k} dz = 2\pi i \bullet b_k \quad (10)$$

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^{n} Res\left[f(z), z_k\right] \quad (11)$$

Example 1. Compute

$$\int_{|z|=2} z^2 sin \frac{1}{z} dz \quad (12)$$

Solution: *Let*

$$f(z) = z^2 sin \frac{1}{z} \quad (13)$$

So, f has an isolated singularity point at z=0, which is included in the contour. By using the Taylor Series expansion for sin(z), the function can be written as follows,

$$z^2 \sin \frac{1}{z} = z^2 \left(\frac{1}{z} - \frac{1}{3! z^3} + \frac{1}{5! z^5} - \cdots \right) = z - \frac{1/6}{z} + \cdots \quad (14)$$

So, Res(f,0) = $b_1 = -\frac{1}{6}$. According to the residue theorem,

$$\int_{|z|=2} z^2 \sin \frac{1}{z} \mathrm{d}z = 2\pi i \cdot Res\left[f(z), 0\right] = -\frac{1}{3}\pi i \quad (15)$$

Example 2. Evaluate

$$\int_C \frac{dz}{z(z-2)^5} dz \quad (16)$$

Where C is the circle $|z-3| = 1$ in the positive sense.

Solution: Let

$$f(z) = \frac{1}{z(z-2)^5} \quad (17)$$

There are 2 singular points at z=0 and z=2. Since only z=2 is included in the contour, the residue at z=2 needs to be calculated.

By using the geometric series,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \left(|z| < 1\right)(18) \quad (18)$$

f can be written as follows,

$$f(z) = \frac{1}{(z-2)^5} \bullet \frac{1}{2 + (z-2)}$$

$$= \frac{1}{2(z-2)^5} \bullet \frac{1}{1 - \frac{-(z-2)}{2}}$$

Then,

$$f(z)=\frac{1}{2(z-2)^5}\sum_{n=0}^{\infty}\left(-\frac{z-2}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^{n-5}\left(0<|z-2|<2\right) \tag{19}$$

Thus, the coefficient of $(z-2)^{-1}$ is the residue of f at z=2.

When n-5=-1, n=4, Res[f(z),2] = $\frac{1}{32}$.

Consequently, by applying the Cauchy's Residue Theorem,

$$\int_C \frac{dz}{z(z-2)^5}dz=\frac{1}{16}\pi i \tag{20}$$

Remark: Firstly, in Example 1 and Example 2 [9], determining whether certain singular points are included in the contour. The function in Laurent Series around each singular point which is included in the contour leads to the residues. By applying the Cauchy's Residue Theorem, functions can be easily integrated, no matter what the denominator, varying from trigonometric equations to polynomials.

2.2 Applications of Cauchy's Residue Theorem

Cauchy's Residue Theorem can be applied to various fields of math, such as finding integrals, as well as constructing Fourier transforms and Laplace transforms [9]. This section mainly displays the usage of Cauchy's Residue Theorem in trigonometric sums.

Theorem 3.

[10] Let $P(\theta)$ be a polynomial of degreelessthan2n in cosθ, where y is a real parameter. It is possible to derive the following identity,

$$\sum_{k=0}^{2n-1}\frac{sin(y+\frac{k\pi}{n})P(y+\frac{k\pi}{n})}{cos(y+\frac{k\pi}{n})-cos\theta}=\frac{2nsin2ny\bullet P(\theta)}{cos2ny-cos2n\theta} \tag{21}$$

Remark: Recalling the Cauchy's Residue Theorem, the integral of the function around simple closed contour C equals to the 2 πi multiplying the summation of the residues at different singular points. The advantage of using Residue Theorem is that the right-hand side in Theorem 3 can be treated as the summation of the residues of a function, instead of going through lots of trigonometric transformations in proving this identity. Hence, a complex function can be defined as follows,

$$f(\alpha)=\frac{2nsin2nysin\alpha\bullet P(\alpha)}{(cos2n\alpha-cos2ny)(cos\alpha-cos\theta)},where 0<\theta<2\pi and 0<y<\frac{\pi}{n}.$$

The indented rectangle described in the positive sense with vertices at $(\pm iR)$ *and* $(2\pi\pm iR)$ where $R\in\mathbb{R}$, is represented by the symbol $C=C_R$.

$$\frac{1}{2\pi i}\oint_C f(\alpha)d\alpha \tag{22}$$

Evaluating the integral of this function, an equation can be deduced with the summation of residues on one side which can be found by using limits at singular points and the value of the integral on the other.

To further understand this identity, numerical values can be substituted into Theorem 3. Letting n=2, $P(\theta)=\frac{sin(2\theta)}{sin\theta}$ in Theorem 3,

$$\sum_{k=0}^{2\times2-1}\frac{sin2(y+\frac{k\pi}{2})}{cos\left(y+\frac{k\pi}{2}\right)-cos\theta}=\frac{4sin4y\bullet sin2\theta}{sin\theta(cos4y-cos4\theta)}=\frac{8sin4y\bullet cos\theta}{cos4y-cos4\theta} \tag{23}$$

3. Conclusion

The calculation of integrals by the Residue Theorem has three steps: finding out the singular points, then using Laurent series to evaluate the residues at each singular points and finally multiplying 2 πi to the sum of the residues. The Residue Theorem can simplify the calculation of integrals for analytic functions including singular points inside a simple closed contour. Besides, showcasing the proof of Residue Theorem, this paper includes remarks of the theorem on trigonometric sum identities. With the continuous development of science and technology, the cross-integration of different subjects is becoming more and more common. How to combine the Residue's Theorem with other fields to solve practical problems is a direction worth exploring. For example, in the field of robots and artificial intelligence, how to use theorems to process large-scale data sets and improve the accuracy of algorithms is a research topic with broad prospects. In the future, it's hoped that the residue theorem can promote the development of related disciplines.

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