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Applications of Taylor Series in Approximate

Calculation and Limits

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Abstract:

Taylor series plays a significant role in mathematics, especially in the fields of mathematical analysis, calculus and applied mathematics. It is not only a theoretical tool, but also an indispensable method in practical applications. For instance, it plays an important role in the fields of physics, engineering and economics. Taylor function uses function which is represented by a series of infinite additive terms, which are calculated by the derivative of the function at a certain point. This paper mainly uses literature research method and empirical research method to introduce the basic information and practical application of Taylor series. Some methods are given to solve related problems. This article demonstrates a brief introduction of Taylor series, then illustrates the steps of proof for Taylor series and provides the solutions of several practical examples. In the end, the paper focuses on the application of Taylor series on limits and analyzes some typical and interesting examples.

Keywords: Taylor series; applications; approximate calculation; limits

1. Introduction

Taylor series can be found in many areas. Many researchers conducted interesting research on Taylor series. In 2024 Yang, Wang and Li improved the registration and positioning efficiency between hull segment measurement data and CAD model. An analytical solution of point set registration based on Taylor series criteria is presented [1]. In the same year Shen and Wan based on Taylor series expansion optimized floating-point arithmetic, and an accurate floating -point arithmetic machine is designed according to the standard in [2]. Cheng and Liu summarized several expansion forms of Taylor series and the application examples of Taylor series in advanced mathematics. Liquid flow control systems were listed. Finally, the feasibility of the results was verified by numerical simulation [3]. Estimation method with Amplitude ratio average in frequency domain based on Taylor series expansion of different orders was proposed in 2023 [4]. A low latitude polarization method based on Taylor series expansion was proposed by Zhang and his team [5]. In 2023 Wang, Xue and Li published an essay which discussed the different order approximations of Taylor series of effective sound velocity. Especially, the applicable conditions for two kinds of Taylor series approximations are given, and their applicability and approximation accuracy change with increasing height [6]. Wan and He improved the control effect of slip rate, a Taylor series feed forward retardation compensation electron- hydraulic braking system slip rate control strategy was proposed [7]. In 2022 Li, Zhang, Wang et al. proposed a localization algorithm that combines the improved whale optimization algorithm with the Taylor series expansion algorithm [8]. A piecewise Taylor series expansion method was proposed for the numerical solution of the first Fredholm integral equation by Liu and Hang in 2022 [9]. An amplitude-spectral integral difference method based on Taylor series expansion was proposed by Zhang, Zhang, Wang et al. [10].

This paper mainly discusses Taylor series and its wide application. First, the thesis defines Taylor series in Section 2 and proves it theoretically by expanding Taylor series. The relevant examples are supplemented to deepen the understanding. The third section focuses on the importance of Taylor series in practical applications, especially in the calculation of limits and approximations. Finally, the paper summarizes the significance of Taylor's formula and emphasizes its important role in the field of mathematics.

2. Taylor Series

Taylor series uses infinite term additive which determined by the derivative of the function at one point to represent a specific function. It was named after the British mathematician Taylor who published Taylor's Formula in 1719. Taylor series, which requires infinite derivability at the point of expansion, is a power series in which a function is expanded into a finite term.

The function of Taylor series:

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \quad (1)$$

2.1 Proof of Taylor Series

The proof contains two parts: the first part is when $z=0$ and another part is when $z \neq 0$.

(1)In the case when $z=0$, the series becomes as follow:

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n(|z^n|<R_0) \quad (2)$$

Then let |z|=r and C_0 represent the positively oriented circle, we can use Cauchy integral formula:

$$f(z)=\frac{1}{2\pi i}\int_{C_0}\frac{f(s)ds}{s-z} \quad (3)$$

when the point z is inside C_0.

Now put $\frac{1}{s-z}$ in the form:

$$\frac{1}{s-z}=\frac{1}{s}\bullet\frac{1}{1-z/s} \quad (4)$$

Replacing z by z/s, rewrite equation as:

$$\frac{1}{s-z}=\sum_{n=0}^{N-1}\frac{1}{s^{n+1}}z^n+z^N\frac{1}{(s-z)s^N} \quad (5)$$

Multiplying by f (s) and integrating each side with respect to s around C_0, we can see that

$$\int_{C_0}\frac{f(s)ds}{s-z}=\sum_{n=0}^{N-1}\int_{C_0}\frac{f(s)ds}{s^{n+1}}z^n+z^N\int_{C_0}\frac{f(s)ds}{(s-z)s^N} \quad (6)$$

After a series of reduction and multiplication we know that:

$$\lim_{N\to\infty}\rho_N(z)=0 \quad (7)$$

To accomplish this, we recall that $|z|=r$ and that C_0 has radius r_0, where $R_0 > r$.

Then, if s is a point on C_0, we can see that

$$|s-z|\geq||s|-|z||=r_0-r. \quad (8)$$

Thus, if M denotes the maximum value of $|f(s)|$ on C_0,

$$|\rho_N(z)|\leq\frac{r^N}{2\pi}\cdot\frac{M}{(r_0-r)r_0^N}2\pi r0=\frac{M_{r_0}}{r_0-r}(\frac{r}{r_0})^N \quad (9)$$

As $\frac{r}{r_0}<1$, the limit above clearly holds.

(2) In the case $z_0 \neq 0$.

Suppose that f is analytic when $|z-z_0|<R_0$. To verify the theorem when the disk of radius R_0 is centered at an arbitrary point z_0, note that the composite function f $(z+z_0)$ must be analytic when $|(z+z_0)-z_0|<R_0$. This last inequality is $|z|<R_0$. If we write $g(z)=f(z+z_0)$, the analyticity of g in the disk $|z|<R_0$ guarantees the existence of a Maclaurin series. Thus, the Taylor series expansion is received.

2.2 Examples

Write sinh z as a taylor series. Let sinh z = −i sin(iz). In order to compute this problem, we only need to recall the expansion which related to Maclaurin series:

$$sinz=\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\quad(|z|<\infty) \quad (10)$$

which becomes:

$$sinhz=\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}\quad(|z|<\infty). \quad (11)$$

3. Application of Taylor Series

Applications of Taylor series can be found in many fields including function approximation, equation solving, physics, engineering, economic modeling. In this essay, we will focus on the application of Taylor series in approximation calculation and limits. Some specific and interesting examples will be presented.

3.1 Application in Approximate Calculation

We can express the function as a polynomial based on the Taylor series to control the error according to the error. The series can be written according to the accuracy requirements. Then we can get the ideal result. Examples of such problem are given below.

Example 1:

(1)Calculate the value of e so that its error is not more than 10^{-6}.

(2)Prove that e is irrational.

Solution:

(1)It is *easytohave* $e=1+1+\frac{1}{2!}+\ldots+\frac{1}{n!}+\frac{e^\theta}{(n+1)!}$, $0<\theta<1$.

Because $0<\theta<1$, $2<e<3$, the error

$$R_9(1)<\frac{3}{10!}=\frac{3}{3628800}<10^{-6}. \quad (12)$$

Hence

$$e\approx 2+\frac{1}{2!}+\ldots+\frac{1}{9!}\approx 2.718281. \quad (13)$$

The error is not more than 10^{-6}.

$$n!\;e-n!(1+1+\frac{1}{2!}+\ldots+\frac{1}{n!})=\frac{e^\theta}{n+1}. \quad (14)$$

If $e=\frac{p}{q}$, the (p,q)=1 is rational number. Take $n \geqslant q$ and $n \geqslant 3$. Then the left side of equation (14) is integral, because $\frac{e^{\theta}}{n+1} < \frac{e}{n+1} < \frac{3}{n+1}$,

When $n > 2$ the right side of equation (14) is not integral. The equation is contradictory. This means e is an irrational number.

Example 2:

(1) Calculate the value of ln2, so that its error does not exceed 10^{-4}.

Solution:

We can consider function

$$f(x) = ln\frac{1+x}{1-x}, -1 < x < 1. \tag{15}$$

Because the Taylor polynomial of order n for ln(1+x) is

$$x - \frac{x^2}{2} + \ldots + \frac{(-1)^{n-1}x^n}{n}, \tag{16}$$

the Taylor polynomial of order n for ln(1-x) is

$$-x - \frac{x^2}{2} - \ldots - \frac{x^n}{n}, \tag{17}$$

Thus, the Taylor polynomial of order 2n for $ln\frac{1+x}{1-x}$ is

$$2(x + \frac{x^3}{3} + \ldots + \frac{x^{2n-1}}{2n-1}). \tag{18}$$

While $f^{(2n+1)}(x) = (2n)!(1+x)^{-2n-1} + (2n)!(1-x)^{-2n-1}$ (19)

$$= \frac{(2n)!}{(1+x)^{2n+1}} + \frac{(2n)!}{(1-x)^{2n+1}}, \tag{20}$$

$$R_{2n}(x) = \frac{1}{2n+1}[\frac{1}{(1+\theta x)^{2n+1}} + \frac{1}{(1-\theta x)^{2n+1}}]x^{2n+1}, \tag{21}$$

Let $\frac{1+x}{1-x} = 2$, x=$\frac{1}{3}$. It leads to $R_{2n}\left(\frac{1}{3}\right) \leqslant \frac{1}{2n+1}\left(\frac{1}{2}\right)^{2n} < 0.0001$

Let n=6, we get

$$ln2 \approx 2(\frac{1}{3} + \frac{1}{3\times 3^3} + \ldots + \frac{1}{11\times 3^{11}}) = 0.6931. \tag{22}$$

Its error is not more than 0.0001(true value is 0.693147180…).

3.2 Application in Limits

Example 3:

Calculate

$$\lim_{x\to 0}\frac{e^{x^2} + 2cosx - 3}{x^4} \tag{23}$$

Solution: Because

$$e^{x^2} = 1 + x^2 + \frac{1}{2!}x^4 + o(x^4) \tag{24}$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^5) \tag{25}$$

Hence

$$e^{x^2} + 2cosx - 3 = (\frac{1}{2!} + 2\cdot\frac{1}{4!})x^4 + o\left(x^4\right) \tag{26}$$

$$\lim_{x\to 0}\frac{e^{x^2} + 2cosx - 3}{x^4} = \lim_{x\to 0}\frac{\frac{7}{12}x^4 + o(x^4)}{x^4} = \frac{7}{12} \tag{27}$$

Example 4:

Find the limit of

$$\lim_{x\to 0}\frac{cosx - e^{-\frac{x^2}{2}}}{x^4} \tag{28}$$

Solution:

As $x \to 0$, The denominator x^4 is the fourth infinitesimal of x,

Thus $cosx$ and e^4 can be expanded to Maclaurin's formula respectively

Because

$$e^x = 1 + x + \frac{x^2}{2!} + o(x^2) \tag{29}$$

Hence

$$e^{-\frac{x^2}{2}} = 1 - \frac{x^2}{2} + \frac{1}{2!}(-\frac{x^2}{2})^2 + o(x^4) \tag{30}$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o\left(x^4\right) \tag{31}$$

$$\lim_{x\to 0}\frac{cosx - e^{-\frac{x^2}{2}}}{x^4} = \lim_{x\to 0}\frac{[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)] - [1 - \frac{x^2}{2} + \frac{x^4}{8} + o(x^4)]}{x^4} = \lim_{x\to 0}\frac{\frac{x^4}{4!} - \frac{x^4}{8} + o\left(x^4\right)}{x^4} = \lim_{x\to 0}[-\frac{1}{12} + \frac{o\left(x^4\right)}{x^4}] = -\frac{1}{12} \tag{32}$$

4. Conclusion

Taylor series has great significance in fields such as mathematics, physics, and engineering. By expanding a complex function into an infinite polynomial, it allows for the approximation of a function near a specific point. This approach not only simplifies the analysis and calculation of functions but also provides a powerful tool for numerical analysis. Taylor series is widely used in solving differential equations, optimizing problems, and signal processing. It plays a key role in physics, engineering, computer science, and financial mathematics. It is used for option pricing and risk management. Additionally, Taylor series aids students in understanding function behavior and trends in educational settings. In summary, Taylor series is a fundamental tool in theoretical mathematics and an indispensable method in practical applications, supporting the advancement of science and technology. This study deepens the understanding of Taylor series and provides strong theoretical support for further research in related fields. Additionally, the method proposed in the study has practical value, particularly in precise calculation and error control. Future research could explore the application of Taylor series on other complex systems, especially in interdisciplinary extensions and algorithm optimization.

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