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Group Theory and Solutions of the Rubik's Cube

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Abstract:

The paper explores the application of group theory to solving the Rubik's Cube. Group theory, a branch of abstract algebra, studies sets equipped with an operation that satisfies specific properties: closure, associativity, identity, and inverses. The Rubik's Cube is treated as a group because its set of moves forms a structure that aligns with these properties. By representing the cube's various configurations and moves through the mathematical constructs of groups, the paper analyzes how the elements transform during cube manipulations. Key concepts such as symmetric groups, homomorphisms, alternating groups, and disjoint cycle decomposition are used to break down the cube's complexity. Additionally, the study demonstrates how group actions, particularly the parity of permutations and orientation sums, govern valid cube configurations. The paper also references methodologies from other studies to apply and refine these mathematical approaches for systematically solving the Rubik's Cube. Through these methods, the research illustrates how the cube can be navigated using logical algorithms grounded in group theory principles

Keywords: Group theory, Rubik's cube, Solutions

1. Introduction

Group theory is a branch of mathematics subject called abstract algebra which studies the elements, properties, and structures of certain defined sets called groups[5]. A group is a set of elements combined with a given operation to satisfy the following four properties: Closure, Associativity, Identity Element, and Inverse Element[4]. These properties will be further explained in detail later in this paper.

A Rubik's Cube is a 3D combination puzzle that consists of a cube with six faces, each made up of nine squares arranged in a three times three grid. The six faces are colored with white, red, green, blue, yellow, and orange. Each face can rotate independently, allowing the smaller squares to be in various arrangements. A Rubik's Cube is considered "Completed" after all the colors in the same face match the color of the centerpiece of the face. [1]

2. Group Theory

2.1 Group Properties The Rubik's Cube satisfies the four fundamental properties of a group:

Closure: Any sequence of rotations results in another valid configuration.

Identity: The identity element corresponds to leaving the cube unchanged.

Inverses: Each move has an inverse that undoes it.

Associativity: The order of applying the moves doesn't change the final configuration.

2.2 Symmetric Group The symmetric groups S_8 (for corner cubies) and S_{12} (for edge cubies) describe the permutations of these cubies when the cube is manipulated. Elements of these groups can represent a Rubik's Cube configuration.

2.3 Homomorphism A homomorphism from the group of cubes moves to S_8 or S_{12} captures how a move affects the corner and edge cubies. These homomorphisms help analyze the structure of the cube's possible configurations.

The sign homomorphism maps permutations $\{\pm 1\}$, which indicates whether a permutation is even or odd. This concept ensures that the parity of the permutations of the corner and edge cubies matches invalid configurations.

2.4 Alternating group The alternating group A_n contains only even permutations and is relevant to the Rubik's Cube because each face twist is an even permutation. The set of all cube moves forms a subgroup of A_{20}, where $n = 8$ (corners) plus $n = 12$ (edges).

2.5 Disjoint Cycle decomposition For functions that go in a loop and eventually end up in the beginning value, we call this kind of function cycle. Also, the numbers from the

cycles are called support

Two cycles are called disjoint if they have no numbers in common, that is $supp\sigma \cap supp\tau$ for cycles σ and τ.

3. Method by Janet Chen

In Janet Chen's paper, the moves of the Rubik's cube are converted to a modified cycle notation and describe where each cubie moves and where each face of the cube moves An example provided showed that if the Rubik's cube is rotated as below,

	f	f	f	
l	d	d	d	r
l	d	d	d	r
l	d	d	d	r
	b	b	b	

	l	l	l	
b	d	d	d	f
b	d	d	d	f
b	d	d	d	f
	r	r	r	

One face was expanded and rotated 90 degrees clockwise. Looking at the cubies, the dlf cubicle is now in dfr cubicle, and this is one part of the cycle. If the same operation is done to the edge cubbies, we'll have $D(dlfdfrdrbdbl)(dfdrdbdl)$ and complete the cycle, and same would apply to other faces.

Then, A Rubik's Cube configuration can be described by:

Positions of corner cubies: Represented by an element σ of S_8, showing how corner cubies move from the start position.

Positions of edge cubies: Represented by an element τ of S_{12}, showing how edge cubies move from the start position.

Orientations of corner cubies: Each corner cubie has 3 possible orientations (0, 1, 2), with 0 being the starting orientation. These are numbered based on the clockwise rotation from a fixed face on each cubicle. This system organizes both position and orientation using group theory elements and consistent numbering.

Now, we look at the face from the top, the cubies would look like

	f	f	f	
l	d	d	d	r
l	d	d	d	r
l	d	d	d	r
	b	b	b	

Thus, cubicle numbering we see would look like

	6		7	
	5		8	

and the cubie labels would look like

	2		1	
1	0		0	2
2	0		0	1
	1		2	

To analyze the move $[D,R] = DRD?^{1} R?^{1}$ of the Rubik's Cube, we track the cube's configuration through four elements: $\sigma, \tau, x, andy$.

τ is a permutation of the 12-edge cubies. For $[D,R]$, it moves the cubie from df to dr, dr to br, and br to df, giving us $\tau = (dfdrbr)$.

σ describes the permutation of 8 corner cubies. The move $[D,R]$ switches dfl with dfr and drb with bru, resulting in $\sigma = (drbbru)(dfldfr)$.

x is the 8-tuple representing corner cubie orientations. After $[D,R]$, the orientations of certain cubies change. For example, $x_3 = 2$ because the b face of drb moves to the u face of ubr, yielding $x = (0,0,2,0,0,2,0,2)$.

y is the 12-tuple representing edge cubie orientations. Since $[D,R]$ only affects edge cubies df, dr, and br, and leaves their orientations unchanged,

y = $(0,0,0,0,0,0,0,0,0,0,0,0)$.

This method of separating the configuration into σ, τ, x, and y helps identify patterns in cube movements significantly.

A Rubik's Cube configuration, $C = (\sigma, \tau, x, y)$, changes when a move $M \in G$ is applied, resulting in a new configuration $C \cdot M$. Performing two moves $M_1 and M_2$ successively gives $(C \cdot M_1) \cdot M_2$, which is the same as $C \cdot (M_1 M_2)$ showing that the moves follow the group action property. The identity move e leaves the configuration unchanged, so $C \cdot e = C$

This illustrates a group action, where elements of a group (cube moves) act on a set (cube configurations).

The valid configurations of the Rubik's cube can be characterized by the theorem that states a configuration (σ,τ,x,y) is valid if and only if $sgn(\sigma)=sgn(\tau)$, $x_i \equiv 0(mod3)$, and $\sum y_i \equiv 0(mod2)$. To prove this, we first show that if (σ,τ,x,y) is valid, these conditions hold. The key idea is that the group acts on the set of Rubik's cube configurations, and valid configurations form a single orbit of this action. If two configurations are in the same orbit, the product of their corner and edge cubie signatures remains constant. This implies that for any valid configuration, $sgn(\sigma)=sgn(\tau)$, as they must match the starting configuration $(1,1,0,0)$. Additionally, another lemma shows that the sums of the corner and edge cubie orientations modulo 3 and 2, respectively, are preserved by basic moves. Thus, if a configuration is valid, it must satisfy the orientation conditions, implying $x_i \equiv 0(mod3)$, and $y_i \equiv 0(mod2)$. This proves one direction of the theorem.

For the converse, assuming $sgn(\sigma)=sgn(\tau)$, $x_i \equiv 0(mod3)$, and $y_i \equiv 0(mod2)$, we aim to show that the Rubik's cube can be solved from such a configuration. The proof involves demonstrating that there exists a series of moves that transforms any configuration satisfying these conditions into the solved state.

First, we show that we can place all corner cubies in the correct positions using a move that adjusts corner cubie positions while preserving their orientations and the edge cubies. Next, we show that corner cubie orientations can be corrected by using moves that affect only two cubies' orientations at a time. Similarly, we handle the edge cubies, first fixing their positions without disturbing the corners, and then adjusting their orientations using appropriate moves. In each case, the key idea is to use group-theoretic properties and conjugation to construct moves that progressively solve different parts of the cube while preserving previously solved parts. Finally, when all these steps are complete, the configuration must be in the solved state, proving the theorem fully.

4. Method by Professor W.D. Joyner

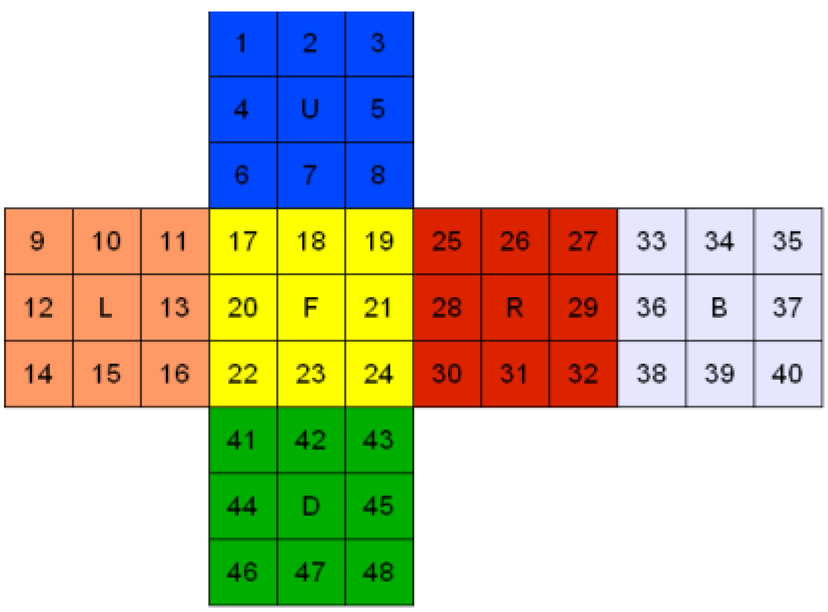
In this method, the below operations are defined:

X: Turn a face 90° clockwise (e.g., Uclk for the top face).

X^{-1}: Turn a face 90° counterclockwise (e.g., Dclk for the bottom face).

X*Y: Perform sequence X followed by Y in that order.

From the start, the Rubik's Cube will be labeled as follows

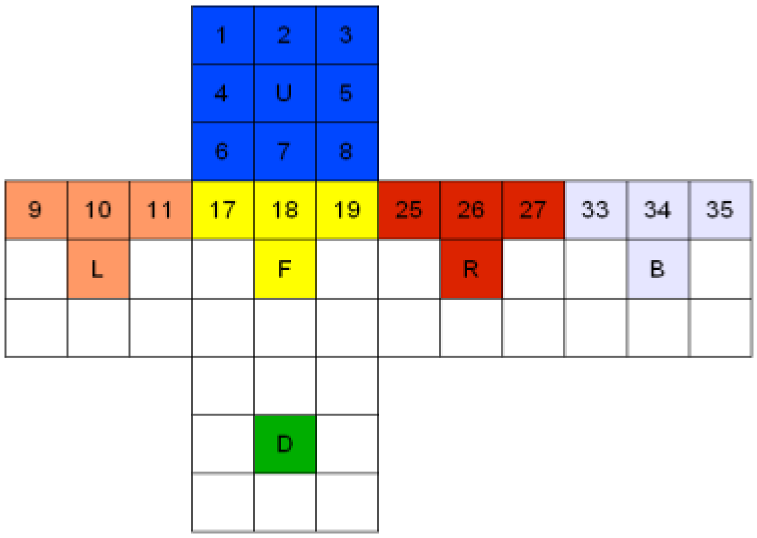


However, in this case, the group generated by the Rubik's Cube permutations has a size of 43 quintillion. Notably, the center facets of each face (U, L, F, R, B, D) remain fixed. The mathematical notations involved in solving include conjugation, commutators, and repeated applications of group elements. Solving the cube follows three stages, with Level 1 focusing on solving the upper face and surrounding edges (21 facets). Key moves include:

1. "Monotwist": $[F,R^{-1}]$ involves specific face rotations.
2. "Monoswap": $D*D^2*D^{-1}$ applies a sequence of face twists.
3. "Monoflip": $(\epsilon R)^4$, a middle slice rotation using

$U^{-1}, D^{-1}, and R$.

4. "Edgeswap": U² exchanges the edges using L, R, and U rotations.



As shown in the picture, this shows the cubies that are being operated in level 1

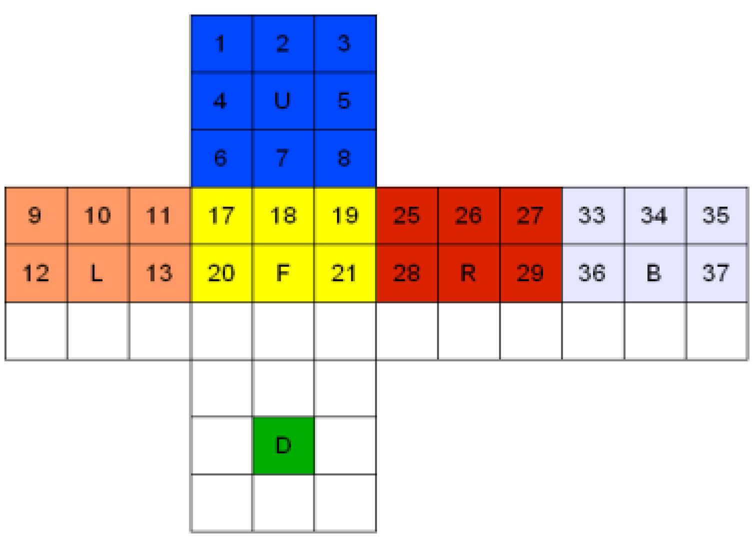
In Level 2 of solving the Rubik's Cube, the goal is to solve the middle band of 12 facets while preserving the results from Level 1. The "clean edge" moves used in this stage are:

Top edge 3-cycle: $R^2 * U * F * B^{-1} * R^{-2} * F^{-1} * B * U * R^2$, which cycles the top edges (uf, ub, ur).

Edge flip without permutation: $[U * F^{-1} * R] * [U^{-1} * B * R^{-1}]$, which flips but does not permute the top edges uf and ub.

Edge pair permutation: $(R^2 * U^2)^3$, which permutes two pairs of edges (uf, ub) and (fr, br).

Top edge pair permutation: $(L^2 * F^2 * B^2 * R^2 * F^2 * B^2)^{(D*B^2*F^2)}$, permuting the top edge pairs (uf, ul) and (ur, ub).



These will be the cubies that are set location after Level 2, and we'll move on to the last level

In Level 3 of solving the Rubik's Cube, the aim is to solve the down face and its surrounding 21 facets while preserving the results from Levels 1 and 2. The following "clean corner" and "clean corner-edge" moves are used:

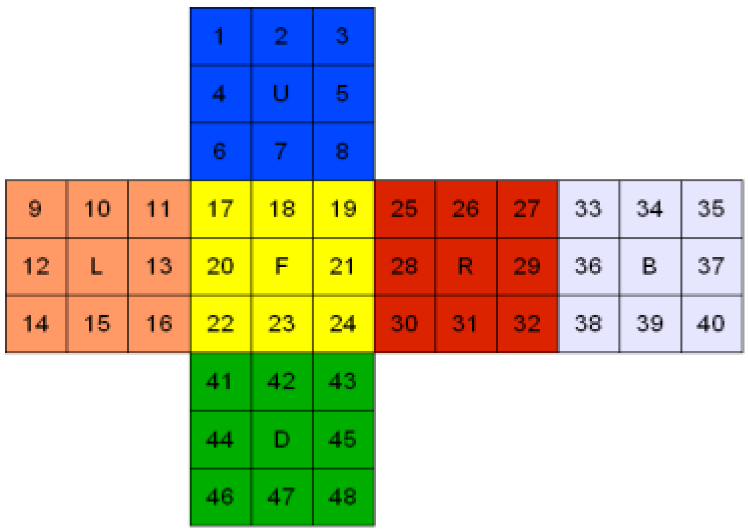
Corner twist: $((D^2)*R*(U^2)*B^2)^2$ twists the ufr corner clockwise and the bld corner counterclockwise.

Alternative corner twist: $(U^2*(D^2)^{(FR^{-1}R)}*U^2)^2$ also twists two corners ($ufrandbld$) similar to the previous move.

Corner pair permutation: $((D^2)^{(FD^{-1}R)}*U^2)^2$, permutes two pairs of corners (ufr,ufl) and (ubr,ubl).

Corner 3-cycle: $[D^{-1}*R,U^{-1}]$ cycles three corners (bru,blu,brd).

Edge and corner permutation: $B^{(U^{-1}*F)}*U^2*B^{-1}*U*B$, permutes two top edges and two top corners (ulb,urb) and (ub,ur).



Thus, by following the cycle, the Rubik's cube can be solved through the three levels. This is applicable to all circumstances of scramble of the Rubik's cube

5. Method By Lindsey Daniels

In this method, the Rubik's Cube will be defined as a group and explore associated theorems and applications.

Clockwise turns are made as if the solver is looking at that face. The inverse of each move is the 90-degree counterclockwise rotation, denoted M_i^{-1} for $M_i \in \{U,F,L,R,B,D\}$ For example, the combination FLU results in turning the front face 90 degrees, then the left face, and finally the upper face. The inverse would be $U^{-1}F^{-1}L^{-1}$.

The Rubik's Cube can be described as a permutation of its 54 facets. Thus, the Rubik's Cube group is a subgroup of a permutation group S^{54}. The permutation group $G=\langle F,L,U,D,R,B\rangle \subset S_{54}$ is called the Rubik's Cube Group. Then, there are two classifications of the Rubik's Cube Group: Legal and illegal. Legal groups only includes moves allowed by the cube's mechanics.

Not all permutations are possible on the Rubik's Cube due to the fixed center facets and the requirement that corner facets only occupy corner positions, and edge facets only occupy edge positions.

Now we look at cubes, each corner cube has three facets, and there are eight corner cubes in total. The orientation of these facets can be described by the cyclic group C_3 . Thus, the orientation of the corner cubes is given by C_3^8 . Thus all the position of all corner facets can be described by the group $C_3^8 \rtimes S_8$.

Every edge cube consists of two facets, with 12 edge cubes in total. The orientation of edge cubes is described by the cyclic group C_2, leading to C_2^{12}. Thus all the positions of all edge facets can be described by the group $C_2^{12} \rtimes S_{12}$.

Each facet's position is assigned a number corresponding to a fixed orientation system. For example, an edge cube starting with number 1 may change its orientation number based on the moves performed. Also, the orientation number for any facet is determined by its position relative to the fixed numbering.

Now let's look at the illegal groups, the Illegal Rubik's Cube Group allows rearranging the cube's facets arbitrarily. However, not all orientations are physically achievable. The Illegal Rubik's Cube Group is $I=(C_2^{12}\rtimes S_{12})\times(C_3^8\rtimes S_8)$

From above, we know that a configuration of the Rubik's Cube is solvable if:

1. $sgn(r)=sgn(s)$ (equal parity of permutations)

2. $v_1+v_2+\ldots+v_8\equiv 0(mod\,3)$ (conservation of twists)

3. $w_1+w_2+\ldots+w_{12}\equiv 0(mod\,2)$ (conservation of flips)

An operation on the cube is valid if:

1. The total number of edge and corner cycles of even length is even.

2. The number of corner cycles twisted right equals the number twisted left (mod 3).

3. There is an even number of reorienting edge cycles.

Using the criteria from the Fundamental Theorems, we can define a reduced group G_0 from the Illegal Rubik's Cube Group, leading to an isomorphism:

$G_0\cong(C_3^7\rtimes S_8)\times(C_2^{11}\rtimes S_{12})$

The order of G_0 is given by:

$$|G_0|=|S_8|\cdot|S_{12}|\cdot C_2^{11}\cdot C_3^7=8!\cdot 12!\cdot 2^{11}\cdot 3^7.$$

By employing these mathematical techniques and understanding the underlying group theory, solvers can systematically navigate the complexities of the Rubik's Cube, achieving the solved state through logic al and structured algorithms.

6. Conclusion

In this paper, the application of group theory to the Rubik's Cube was explored. By modeling the Rubik's Cube as a group with its moves as elements, the cube's compliance with group axioms such as closure, identity, inverses, and associativity was confirmed. The concept of group actions was used to explain how sequences of moves can systematically transform cube configurations, emphasizing the importance of maintaining consistent permutation parity and orientation sums. The alternating group and sign homomorphism explained the mechanism of the cube's moves.

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