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Application of Residue Theorem in Calculating Improper Integral in Calculus

Zeyu Wang

School of RDFZ Xishan University, Beijing, China
Corresponding author: 202003023330@stu.sdp.edu.cn

Abstract:

In this paper, the residue theorem and its applications in the field of calculus are discussed. This study first emphasizes the importance of the residue theorem in the analysis of complex functions, and points out its wide applicability in various branches of mathematics. In the second part, this article explores some basic concepts in depth, including a section on singularities and higher-order singularities, precise definitions of residuals, and different methods used to calculate them. Building on this theoretical foundation, this paper transitions to practical applications, showing how to use the residual theorem to compute integrals, with particular emphasis on its ability to simplify calculations and provide efficient solutions to other thorny problems. Through a series of detailed examples, the validity of the theorem in solving difficult integrals solved by standard methods is emphasized. The conclusion reiterates the key role of the residue theorem in calculus and emphasizes its indispensable contribution to calculus calculation by providing accurate and simplified solutions.

Keywords: Residue theorem, Calculus, Complex function, Improper integral.

1. Introduction

The residue theorem is a fundamental concept in complex analysis, known for its ability to simplify the calculation of complex integrals, especially in cases where the integral path encounters singularities on the complex plane. This theorem is derived from the analysis of the behavior of analytic functions near isolated singularities. It assumes that if a function is analytic within and on a closed path, and there are a finite number of isolated singularities within that path, then the integral along that path can be determined by summing the residues at those singularities. By reducing the problem to the calculation of residuals, the residuals theorem transforms complex path integrals into simpler calculations, making it an indispensable tool in the field [1]. The residual theorem allows challenging problems to be solved more efficiently. By converting complex integrals into residual calculations, the theorem Bridges the gap between pure and applied mathematics, demonstrating its versatility and practicality in various mathematical environments.

A major advantage of the residual theorem is its ability to deal with integrals that are difficult to deal with using traditional techniques. For example, integrals involving trigonometric functions, oscillating integrals, and integrals over infinite intervals can often be effectively solved by using the residual theorem [2]. This approach not only simplifies the calculation process, but also provides a systematic approach to solving problems that are otherwise difficult to manage. The ability to choose the appropriate integration path and focus on residuals at singularities allows mathematicians to reduce complex integrals to more straightforward algebraic operations, demonstrating the powerful utility of the residual theorem [3]. In addition, the residue theorem plays a crucial role in connecting the various branches of mathematics. Its application is not limited to calculus, but extends to fields such as physics, engineering, and number theory, where complex integrals often occur. In these disciplines, the residual theorem is used to solve problems involving electromagnetic fields, quantum mechanics, and fluid dynamics. The theorem's ability to simplify and solve complex integrals in these different domains underscores its importance as a versatile and powerful analytical tool [4].

The rest of the paper is organized as follows. In Section 2, It explains the residue theorem and some basic concepts. In Section 3 it shows some examples of calculus using the residue theorem. In the end, the function of residue theorem in calculus is summarized.

2. Theory and Method

2.1 Concepts and Residue Theorems

Before introducing the residue theorem, it is helpful to introduce some concepts. The residue is the coefficient of the negative first power of the Laurent series at an isolated singularity inside a closed curve. A complex function is a function in which both the independent and dependent variables are complex. This is the point at which the function of complex variables cannot be resolved (i.e., cannot be defined or the value is infinite) at some point. Common singularities are poles and essential singularities.

If $f(z)$ is a function that is analytic on a complex plane except for some isolated singularities, and C is a simple closed contour encircling these singularities in a counterclockwise direction, then the contour integral of $f(z)$ along C can be calculated by summing the residues at these singular points [5]

$$\oint_C f(z)dz = 2\pi i \sum Res(f, z_k)$$

where $Res(f, z_k)$ represents the residue of $f(z)$ at the singularity z_k.

2.2 Applications of the Residue Theorem

There are many applications of Residue Theorem. Firstly, it can be used to calculate complex real integrals. One can use contour integrals in the complex plane to compute real integrals that are difficult to solve directly. Secondly, it can analyze function properties. The residue helps quickly determine certain characteristics of a function, such as whether it has a pole at a point and the order of the pole. It also has applications in physics. The Residue Theorem is frequently used to solve complex problems involving complex numbers in fields such as electromagnetism, quantum mechanics, and other areas of physics. Next the passage will talk about the applications of the Residue Theorem in the Calculus. The residue theorem is a very useful tool for some calculus calculations. The residue theorem can often be applied to calculations of some calculus. These integrals are 1) Complex integral on a closed path, 2) Real integrals over infinite intervals, 3) Definite integrals with trigonometric functions, 4) oscillatory integrals, 5) Lorentz integrals, and 6) Real definite integral with singularities [6].

Before the passage are more in-depth discussion of the application of the residue theorem to these integrals. In this section the author will explain the two concepts of the Calculus. The first is Singularities of first order (or simple poles). A simple pole is a singularity where the function $f(z)$ has a pole of order one at a point z_0. This means that the function $f(z)$ near z_0 can be expressed as:

$$f(z) = \frac{g(z)}{(z - z_0)} \tag{2}$$

where $g(z)$ is analytic near z_0 and $g(z_0)$ is non-zero. The residue of the function $f(z)$ at the simple pole z_0 can be calculated as $g(z_0)$. The author shall make an example. The function $f(z) = \frac{1}{z}$ has a simple pole at $z = 0$ because it can be written as $\frac{1}{z}$. The second is Higher-Order Pole. A higher-order pole is a singularity where the function $f(z)$ has a pole of order greater than one at a point z_0. In this case, the function $f(z)$ near z_0 can be expressed as: $f(z) = \frac{g(z)}{(z - z_0)^n}$ where $n > 1$, and $g(z)$ is analytic near z_0 with $g(z_0)$ non-zero. The integer n indicates the order of the pole. The example is the function $f(z) = \frac{1}{(z-1)^2}$. It has a second-order pole at $z = 1$ because it can be written as $\frac{1}{(z-1)^2}$. That is all the concept in this part, and next part will focus on the Results and Applications.

3. Results and Application

3.1 Complex Path Integral and Analytic Extension

The core of the residue theorem is that it relates a complex path integral to the residue of a singularity surrounding the interior of the path. When dealing with complex path integrals, the problem can be greatly simplified by properly selecting the integral path (such as extending the integral path to the complex plane). This continuation usually involves choosing a semicircular path in the upper or lower half plane so that the integral outside the path tends to zero, thereby focusing the difficulty of integration on the singularity inside the path.

Supposed that the author wants to calculate the following integral [7]

$$\oint_C \frac{dz}{z(z-1)} \tag{3}$$

where path C is around the origin and simple closed path for $z=1$

The first thing is to identify the singularity, which is $z=0$ and $z=1$, the second step is to Calculating residue. When $z=0$, the residue is $Res\left(\frac{1}{z(z-1)},0\right)=\frac{1}{1}=1$. When $z=1$, the residue is $Res\left(\frac{1}{z(z-1)},1\right)=\frac{1}{-1}=-1$. By using the residue theorem, it is found that

$$\oint_C \frac{dz}{z(z-1)}=2\pi i\left(1+(-1)\right)=0 \tag{4}$$

So, follow the closed path, the integral value of C is 0. The residue theorem is well suited for dealing with integral problems on closed paths involving complex functions, especially when the function has a finite number of isolated singularities

3.2 Dealing with Integrals Over Infinite Intervals

The residue theorem is particularly powerful when dealing with integrals of infinite intervals. By extending the integral path to the complex plane and using the semicircular path (whose radius tends to infinity), some real integrals that are difficult to compute directly can be converted into complex integrals that are easy to compute.

Supposed that the author wants to calculate the following integral [8]

$$I=\int_{-\infty}^{\infty}\frac{dx}{x^2+1} \tag{5}$$

The first step extend to complex integrals will expand x to a complex variable z, and consider complex functions $f(z)=\frac{1}{z^2+1}$ .Next the author will choose a closed path C_R of radius R consisting of a line segment from $-R$ to R on the real axis and an upper semi-arc. Third step is to compute complex integral. Since the function $f(z)$ has second-order poles at $z=i$ and $z=-i$, one only needs to consider the singularity $z=i$ of the upper half plane. The residue of the function $f(z)$ at $z=i$ is

$$Res(f,i)=\lim_{z\to i}(z-i)\,f(z)=\lim_{z\to i}\frac{z-i}{(z-i)(z+i)}=\frac{1}{2i} \tag{6}$$

According to the residue theorem, the complex integral is:

$$\oint_{C_R} f(z)dz=2\pi i\cdot\frac{1}{2i}=\pi \tag{7}$$

As R approaches infinity, the integral over the semi-arc approaches zero, thus

$$I=\int_{-\infty}^{\infty}\frac{dx}{x^2+1}=\pi \tag{8}$$

By calculation the real integral I has the value π.

By using the residue theorem, real integrals on certain infinite intervals can be converted to complex integrals for computation. This method is particularly suitable for integrals that can find an appropriate closed path in the complex plane and apply the residue theorem.

3.3 Trigonometric Integrals and Complex Exponential Functions

For integrals involving trigonometric functions, an important application of the residue theorem is to express trigonometric functions as complex exponential functions, thereby transforming the integration problem into the calculation of integrals over complex functions. Especially when the integration involves periodic functions, the calculation process can be simplified by choosing a suitable integration path in the complex plane (such as a path around the unit circle).

The author aims to consider calculating the following points [9]

$$I=\int_0^{\infty}\frac{sin(x)}{x}dx. \tag{9}$$

The first is to convert trigonometric functions to complex functions use $sin(x)=\frac{e^{ix}-e^{-ix}}{2i}$, and the Eq. (9) turns to be

$$I=\int_0^{\infty}\frac{\frac{e^{ix}-e^{-ix}}{2i}}{x}dx. \tag{10}$$

Then, one can divide it into two integrals $I=\frac{1}{2i}(\int_0^{\infty}\frac{e^{ix}}{x}dx-\int_0^{\infty}\frac{e^{-ix}}{x}dx)$. After this one should select complex path. Consider complex functions $f(z)=\frac{e^{iz}}{z}$ integral in the complex plane. This paper chooses a closed path C that includes integrals on the real axis and integrals on a semi-arc C_R. Using the integral path C_R, it consists of the interval $[-R,R]$ on the real axis and the semi-arc $|z|=R$. Next, the author calculates the residue, in which $f(z)=\frac{e^{iz}}{z}$ has a pole of first order at z. $Res\left(\frac{e^{iz}}{z},0\right)=\lim_{z\to 0}e^{iz}=1$. According to the residue theorem $\oint_C\frac{e^{iz}}{z}dz=2\pi i\cdot 1=2\pi i$. As $R\to\infty$, the integral over the semi-arc tends to zero. Therefore, only the integrals on the real axis should be considered in the calculation: $\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx=2\pi i$. Using symmetry, one can get $\int_{0}^{\infty}\frac{e^{ix}}{x}dx=\frac{1}{2}\left(\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx\right)=\frac{\pi i}{2}$, and the result of $\int_{0}^{\infty}\frac{e^{-ix}}{x}dx$ is $-\frac{\pi i}{2}$. Consequently, $I=\frac{1}{2i}\left(\frac{\pi i}{2}-\left(-\frac{\pi i}{2}\right)\right)=\frac{\pi}{2}$, and the final solution is the $\frac{\pi}{2}$.

By converting trigonometric functions to complex functions, choosing the appropriate complex path, calculating residues, and dealing with additional parts on the path, people can effectively apply the residue theorem to compute definite integrals involving trigonometric functions. This method is especially useful when dealing with integrals over infinite intervals.

3.4 Oscillation Integration and Analytical Extension

When the definite integral involves poles (i.e., the singularity of the function lies at or inside the endpoint of the integral interval), direct computation can be difficult. In this case, using the residue theorem and Cauchy principal value, the integral can be efficiently computed by bypassing the path to the poles.

Supposed that the author wants to calculate the following oscillation integral [10]

$$I=\int_{-\infty}^{\infty}\frac{e^{ix}}{x^2+1} \qquad (11)$$

It can consider extending this integral to the complex plane to define complex functions $f(z)=\frac{e^{iz}}{z^2+1}$ and close the integral path in the upper half plane. $f(z)$ has a simple pole at $z=i$.(Since z^2 +1=0 is true at $z=i$). This can be done by calculating the residue at the pole by the residue theorem

$$Res(f,i)=\lim_{z\to i}(z-i)\frac{e^{iz}}{z^2+1}=\lim_{z\to i}\frac{e^{iz}}{2i}=\frac{e^{-1}}{2i} \qquad (12)$$

Thus, using the residue theorem, the integral on the closed path is:

$$\oint_C f(z)dz=2\pi i\cdot\frac{e^{-1}}{2i}=\pi e^{-1} \qquad (13)$$

So, the solution is $I=\pi e^{-1}$.

The application of the residue theorem in oscillatory integration is to extend the integral to the complex plane and use the residue of complex functions to calculate the originally complex real integral. This method is especially suitable for the case where there is a fast oscillation factor in the integrand and can simplify the calculation process effectively.

3.5 Lorentz Integrals

Lorentz integrals are closely related to Lorentz distributions and are widely used to describe resonance phenomena, spectral line widths, and probability distributions in statistics. The Lorentz distribution is a typical probability density function, usually used to describe the peak distribution.

Here, the author considers calculating the following Lorentz integral:

$$I=\int_{-\infty}^{\infty}\frac{dx}{x^2+a^2} \qquad (14)$$

First, one can set a complex function $f(z)=\frac{1}{z^2+a^2}$. The function $f(z)=\frac{1}{z^2+a^2}$ has poles at $z=\pm ia$. It needs to calculate the residue of these points, The residue at $z=ia$ is given by

$$Res\left(\frac{1}{z^2+a^2},ia\right)=\lim_{z\to ai}(z-ia)\frac{1}{z^2+a^2}=\frac{1}{2ia} \qquad (15)$$

As the same, the Residue at $-z=ia$ is $-\frac{1}{2ia}$.

According to the residue theorem, the complex integral on a closed path is:

$$\oint_C\frac{dz}{z^2+a^2}=2\pi i\cdot\frac{1}{2\pi a}=\frac{\pi}{a} \qquad (16)$$

As $R\to\infty$, the integral part on the semi-arc approaches zero, so the integral on the real axis is:

$$\int_{-\infty}^{\infty}\frac{dx}{x^2+a^2}=\frac{\pi}{a} \qquad (17)$$

The residue theorem provides an effective tool for calculating Lorentz integrals. By extending the Lorentz integral to the complex plane, choosing a suitable closed path, calculating the residue of the complex function, and dealing with the additional parts on the path, the exact value of the Lorentz integral can be obtained. This method is especially useful when dealing with integrals over infinite intervals.

3.6 Dealing with Definite Integrals with Poles

When the definite integral involves poles (i.e., the singularity of the function lies at or inside the endpoint of the integral interval), direct computation can be difficult. In this case, using the residue theorem and Cauchy principal value, the integral can be efficiently computed by bypassing the path to the poles.

The author considers computing the following real definite integral with singularities

$$I=\int_{-\infty}^{\infty}\frac{dx}{x^2+1}. \tag{18}$$

The integrand of this integral has a singularity $z=\pm i$ in the complex plane. Firstly, one can extend the real function $\frac{1}{x^2+1}$ to the complex function $f(z)=\frac{1}{z^2+1}$. There are poles at $z=\pm i.$ Then one should calculate the residue of a complex function. The residue at $z=i$ is $Res\left(\frac{1}{z^2+1},i\right)=\lim\limits_{z\to -i}(z-i)\frac{1}{z^2+1}=\frac{1}{2i}$.

Likely, the Residue at $z=-i$ is $-1/2i$, and in practice one just needs to use the singularity of the upper half plane, $Res\left(\frac{1}{z^2+1},i\right)=1/2i$. According to the residue theorem, the complex integral on a closed path

$$\oint_C\frac{dz}{z^2+1}=2\pi i\cdot\frac{1}{2i}=\pi \tag{19}$$

Thus, the result is π.

By extending the integral to an integral on the complex plane, choosing an appropriate closed path, calculating the residue of the complex function, and dealing with the integral on a semi-arc, people can get the desired real integral result. This method is especially useful for dealing with integrals involving polynomials in the denominator.

4. Conclusion

The residue theorem is a basic tool in complex analysis, which turns complex integral problems into residue finding tasks and greatly simplifies various types of integral calculations. It is particularly valuable in dealing with integrals over complex planes, real integrals over infinite intervals, definite integrals involving trigonometric functions, oscillating integrals, and integrals with singularities. The power of the theorem lies in its ability to reduce complex integrals to simpler algebraic operations by carefully choosing a suitable path and utilizing residues on singularities. This method is not only of great significance in theory, but also has practical application value in many mathematical fields including calculus. In order to apply the residue theorem effectively, people must master the basic formula of the residue theorem and skillfully use geometric and algebraic techniques on the complex plane. This article discusses how to choose the appropriate integration path and focus on residuals at singularities, thereby simplifying challenging integrals into more manageable calculations. By transforming complex path integrals into simple residual calculations, the residual theorem demonstrates its wide applicability and practicality in solving complex problems. Through detailed examples, the practical significance of the theorem is emphasized, especially in simplifying the real integral which is difficult to be calculated by traditional methods, thus emphasizing its important role in higher mathematical analysis.

References

[1] Stewart, J. Calculus: Early transcendentals (9th ed.). Cengage Learning, 2020.

[2] Mathews, J. H., & Howell, R. W. Complex analysis for mathematics and engineering (6th ed.). Jones & Bartlett Learning, 2012.

[3] Jerrold, E. M. Advanced calculus: A course in mathematical analysis (2nd ed.). Springer, 2015.

[4] Gamelin, T. W. Complex analysis (Undergraduate Texts in Mathematics). Springer, 2001.

[5] Stein, E. M., & Shakarchi, R. Complex analysis (Princeton Lectures in Analysis, Vol. 2). Princeton University Press, 2010.

[6] YI Cai-feng. The Cauchy Integral Formula and Its Application in Integration. Journal of Jiangxi Normal University (Natural Science), 2010, 34(01):6-12.

[7] He Ming-Xuan. (2021). Application of residue theorem in some real integrals. Mathematics study and research, 2021 (05):138-139.

[8] Gao Yuan-Yue. A brief discussion on the application of residue theorem. The Farmers Consultant, 2017, (20): 129.

[9] Xu J.-Z. Research on application of residue theorem in real integrals. Journal of Xichang University (Natural Science Edition), 2018, 32(03):57-59.

[10] Zeng Qiao. Explore the application of residue theorem to solve different types of integrals. Technology Innovation and Application, 2020, (11):175-176.