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Taylor Formula and Its Application Problem

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Abstract:

Taylor formula and Taylor series are particularly important in mathematical analysis teaching and are tools for numerical calculation. Their applications are very extensive, including calculating limits, indefinite integrals, definite integrals, high - order derivative values, determining the series are convergent or divergent, proving inequalities, approximate treatment, solving differential equations, and proving identities. This paper mainly introduces the definition of Taylor's formula, the formula along with related proof, and the application of approximate calculation, indefinite integral, inequality proof. It was developed in the study of function expansions to provide a fine description of how a function behaves near a certain point. The significance of Taylor's formula is that it provides a way to expand a function into an infinite series, so that the properties of a function near a certain point can be studied. It can increase the quality of teaching as well as broaden the scholastic problem - solving ideas and improving their ability to solve problems.

Keywords: Taylor formula; approximate calculation; inequality

1. Introduction

The history of Taylor's formula can be traced back to the 18th century, by the famous British mathematician Edward Taylor. Previously, since ancient times, people have explored the derivation of calculus for a long time, but there is no effective method to make the calculation easier. Edward Taylor studied the theory of calculus and gradually discovered Taylor's formula, thus solving this difficult problem.

In 2010, Tan gave some examples that introduce Taylor formula and used Taylor series to solve the questions that judge series' divergence, and proved the inequation that related with integration [1]. In 2016, Zhao and his team studied the differentiability of Taylor formula "intermediate point function" and established the first-order differentiability of Taylor formula "intermediate point function" at point a under certain conditions, and illustrated the validity and universality of the results in this paper [2]. In 2016, Li and Zhang used the concept of comparative function to study the asymptotic and differentiable properties of Taylor's formula "intermediate point function". Under certain conditions, he established the first-order differentiability and asymptotic properties of Taylor's formula "intermediate point function" at point A [3]. In 2013, Li gave two kinds of Taylor

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formulas with different remainder are introduced, namely, the Taylor formula of Peano remainder along with the Taylor formula of Lagrange remainder. The calculation and proof of Taylor's formula was given, including the limit operation, the judgment of convergence and divergence of series and generalized integral [4]. In 2019, Liu, Wei, Zong provided a maximum power tracking method based on Taylor formula, system and terminal equipment, using a nonlinear tracking differentiator constructed with saturation function, and using Taylor formula to optimize the phase delay performance of the nonlinear tracking differentiator, in order to solve the contradiction between filtering effect and phase delay effect [5]. In 2015, Han studied the extension of Newton-Leibniz formula and Taylor formula to parametric functions, and used it to determine the quantity of zeros of parametric functions along with the quantity of periodic solutions of differential equations [6]. In 2013, Fan applied Taylor's formula to determine the convergence of alternating ordinal, which not only overcomes the shortcoming that Leibniz criterion cannot determine the non-monotone decline of general terms, but also can be widely used in complex series constructed by composite function [7].

Section 2 introduces the definition of Taylor's formula, the formulas and related proofs, and an example of the expansion of Euler's formula and the Taylor expansion of the inverse trigonometric function. Section 3 shows the application of Taylor formula to approximate calculation and proving inequality.

2. Taylor Theorem

Theorem: Taylor's theorem is a mathematical theorem that describes the relationship between the values of a real or complex function at a point and its values and derivatives of all orders at that point.

Proof: Let be $f(y)$ has n-degree derivation when x= x_0, $p_n(y)=c_0+c_1(y-y_0)+c_2(y-y_0)^2+\cdots+c_n(y-y_0)^n$.

Let be $f(y)-p_n(y)$ *betheinfinitesimalofhigherorder* to $(y-y)^n$

$$f(y)=\ p_n(y)=c_0 \tag{1}$$

$$f'(y_0)=p_n{}'(y_0)=c_1+2c_2(y-y_0)+3c_3(y-y_0)^2+\cdots+nc_n(y-y_0)^{n-1}. \tag{2}$$

When $y=y_0, f'(y_0)=c_1$,

$$f''(y_0)=p_n^{''(y_0)}=2!c_2+3\times2c_3(y-y_0)+\cdots+\text{n(n-1)}(y-y_0)^{n-2} \tag{3}$$

When $y=y_0$, $f''(y_0)\ =2!c_2$, so $c_2=\dfrac{f''(y_0)}{2!}$,

$$f'''(y_0)=3!c_3+4\times3\times2c_4(y-y_0)+\cdots \tag{4}$$

When $y=y_0$, $f'''(y_0)=3!c_3$, so, $c_3=\dfrac{f'''(y_0)}{3!}$,

Continue to calculate until $c_n=\dfrac{f^n(y_0)}{n!}$,

Thus, $f(y)=f(y_0)+\dfrac{f'(y_0)}{1!}(y-y_0)+\dfrac{f''(y_0)}{2!}(y-y_0)^2+\cdots+\dfrac{f^{(n)}(y_0)}{n!}\left((y-y_0)^n\right)+R_n(y)$.

$R_n(y)$ *istheremainderofTaylorformula,wheny* $\rightarrow y_0$. *Iti*sin *finitelysmallerthan* $(y-y_0)^n$.

Example 1. [8] Euler formula is derived by employing the Charon series expansion, $\sin x=\dfrac{e^{ix}-e^{-ix}}{2i}, cosx=\dfrac{e^{ix}+e^{-ix}}{2}$.

Proof: When x ∈ R, it is obtained by the power expansion of the exponential function $e^x=\sum_{n=0}^{\infty}\dfrac{x^n}{n!}, x\in(-\infty,+\infty)$. Replacing the real variable x with a pure imaginary number yields $thate^{ix}=\sum_{n=0}^{\infty}\dfrac{(ix)^n}{n!}=1+ix+\dfrac{1}{2!}(ix)^2+\dfrac{1}{3!}ix^3+\dfrac{1}{4!}(ix)^4+\dfrac{1}{5!}(ix)^5$.

$Becauseofi=\sqrt{-1}, i^{(4n+2)}=-1, i^{(4n+3)}=-i, i^{(4n+4)}=1, n=0,1,2,\cdots$, Thus, $e^{ix}=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)$ $=cosx+i\sin x$.

So, $e^{ix}=cosx+i\sin x$. Denote this display as ①

Substitute $-xforxin$ ①

$$e^{(-ix)}=cosx-i\sin x \tag{5}$$

According to ① and (5), we can get $\sin x=\frac{e^{ix}-e^{-ix}}{2i}, cosx=\frac{e^{ix}+e^{-ix}}{2}$.

Example 2. Find the Taylor expansion of the inverse trigonometric function $\sin^{-1}x$.

Proof:

$$\left(\sin^{-1}x\right)'=\frac{1}{\sqrt{1-x^2}} \tag{6}$$

$=1+\frac{1}{2}x^2+\frac{3!!}{4!!}x^4+\frac{5!!}{6!!}x^6+\cdots$

Hence,

$Sin^{-1}x$

$$=\int\left(1+\frac{1}{2}x^2+\frac{3!!}{4!!}x^4+\frac{5!!}{6!!}x^6+\cdots\right)dx \tag{7}$$

$=x+\frac{1!!}{2!!}\bullet\frac{x^3}{3}+\frac{3!!}{4!!}\bullet\frac{x^5}{5}+\frac{5!!}{6!!}\bullet\frac{x^7}{7}+\cdots$

3. The Application of Taylor Formula

Next, Taylor formula will be employed in approximate calculation. The function of simplifying complexity provides an effective approach for approximating complex function values [9].

Example 3. Calculate the approximate value of $ln1.2$.

Proof: $ln1.2=ln(1+0.2)$,let $f(x)=ln(1+x)$, $andx=0.2$

Derivation:

$f'(x)=\frac{1}{1+x}, f''(x)=-\frac{1}{(1+x)^2}$, $f'''(x)=\frac{2}{(1+x)^3}$, $f''''(x)=\frac{-6}{(1+x)^4}$

$$f'''''(x)=\frac{24}{(1+x)^5}, f''''''(x)=\frac{-120}{(1+x)^6}; \tag{8}$$

Substitute into value:

$f'(0)=1$, $f''(0)=-1$, $f'''(0)=2$, $f''''(0)=-6$, $f'''''(0)=24$,

$$f''''''(\xi)=\frac{-120}{(1+\xi)^6} \tag{9}$$

$(\xi isbetween 0 andx)$

Hence,

$f(0.2)\approx 0+1\times 0.2+\frac{1}{2!}\times(-1)(0.2)^2+\frac{1}{3!}\times 2\times(0.2)^3+\frac{1}{4!}\times(-6)\times(0.2)^4+\frac{1}{5!}\times 24\times(0.2)^5$

$$\approx 0.1823 \tag{10}$$

Example 4. If a, b>0, and a+b=1, Prove that for any x ≥ 0, there are: $e^x \geq (1+ax)(1+bx)$.

Proof:

$$e^{ax}\geqslant ax+1>0 \tag{11}$$

$$e^{bx}\geqslant bx+1>0(12)$$

$$e^{ax}\times e^{bx}\geqslant(ax+1)(bx+1) \tag{13}$$

$$e^{(a+b)x}\geqslant(ax+1)(bx+1) \tag{14}$$

$$e^x\geq(1+ax)(1+bx) \tag{15}$$

Example 5. [9] Calculate an approximation of e accurate to 10^{-4}.

Proof:

$$e^x=1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots(-\infty<x\leqslant+\infty)+\cdots \tag{16}$$

Let be $x=1, e=1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}+\cdots$

If the sum of the first n terms is taken as an approximation of e, the error is

$$|r_n|=\frac{1}{n!}+\frac{1}{(n+1)!}+\frac{2}{(n+2)!}+\cdots \tag{17}$$

$=\frac{1}{n!}\left[1+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\cdots\right]<\frac{1}{n!}\left[1£«\frac{1}{n}+\frac{1}{n^2}+\cdots\right]$

$=\frac{1}{n!}\bullet\frac{1}{1-\frac{1}{n}}$

$=\frac{1}{(n-1)!(n-1)}$.

So, for $|r_n|<10^{-4}$,as long as $\frac{1}{(n-1)!(n-1)}<10^{-4}$,

Empirical calculation, $\frac{1}{6!6}=\frac{1}{4320}>10^{-4}, \frac{1}{7!7}$

$=, \frac{1}{35280}<10^{-4}$,

Thus n=7, $e\approx 1+1+\frac{1}{2!}+\cdots+\frac{1}{7!}$,

$\approx 1+1+0.50000+0.16667+0.04167+0.00833+0.00139$
$+0.00020=2.7183$

Example 6. [10] Proof: When 0<z<1, there has

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$\frac{(1+x)^{1+x}}{(1-x)^{1-x}} < e^{2x} < \frac{1+x}{1-x}$.

Proof: Let' s say $f(u) = lnu.U \sin g$ Taylor's formula, we can get

$$f(u) = f(1)£«f'(1)(u-1)£«\frac{f''(\xi)}{2!}(u-1)^2(\xi isbetween 1 andu).$$

So $f(1+x) = f(1)£«f^{'(1)}x£«\frac{f^{''(\xi 1)}}{2!}x^2(1<\xi_1<1£«x)$,

$$f(1-x) = f(1)£«f^{'(1)(-x)}£«\frac{f^{''(\xi 2)}}{2!}(-x)^2(1-x<\xi_2<1). \quad (18)$$

$So, ln(1+x) = x - \frac{1}{\xi_1^2} \bullet \frac{x^2}{2}, (1<\xi_2<1+x).$

$$ln(1-x) = \text{-} x - \frac{1}{\xi_2^2} \bullet \frac{x^2}{2}, (1-x<\xi 2<1). \quad (19)$$

Therefore, $ln(1+x) -- ln(1-x) = 2x + \left(\frac{1}{\xi_2^2} - \frac{1}{\xi_1^2}\right)\frac{x^2}{2} > 2x.$

Let' s say $g(u) = ulnu.Accordingto$ Taylor's formula, we can get.

$$g(u) = g(1)£«g'(1)(u-1)£«\frac{g^{''(\eta)}}{2!}(u-1)^2, (1<\eta<u). \quad (20)$$

Thus, $(1+x)ln(1+x) - (1-x)ln(1-x)$

$=2x£«\left(\frac{1}{\eta 1} - \frac{1}{\eta 2}\right) < 2x.$

Thus, $ln\frac{(1+x)^{1+x}}{(1-x)^{1-x}} < 2x$.

4. Conclusion

By studying Taylor's formula, the readers will know what Taylor's formula is and what Taylor's formula does. The readers can understand and analyze the properties and behavior of complex functions, especially in cases that require accurate calculations and predictions. Besides, Taylor's formula is related to other concepts in calculus, such as Lagrange's mean value theorem and Cauchy's mean value theorem, which together form the core of calculus. In summary, studying Taylor's formula not only deepens our understanding of the behavior of functions in mathematics and physics, but also provides a powerful tool for approximating the values of complex functions, estimating errors, and making more accurate predictions and calculations In addition, the meaning of Taylor's formula is to build a polynomial by using information about the derivative of a function at a certain point that approximates the value of that function in the vicinity.

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