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# Homotopy Theory and Fundamental Groups in Topological Spaces: Theoretical Foundations and Applications

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**Abstract:**

This paper explores homotopy groups, a central topic in algebraic topology, and provides a detailed examination of their foundational role in understanding the structure of topological spaces. Beginning with the definitions and operations of homotopy groups, the discussion progresses to the construction of a chain complex and a rigorous proof of its exactness. The paper delves into the application of homotopy theory to fibrations, deriving an exact sequence that elucidates the intricate relationship between loop spaces and the fundamental group in simply connected spaces. This relationship underscores how fundamental groups can be interpreted through the lens of homotopy, particularly in the context of loop spaces. The theoretical results are further applied to algebraic varieties and schemes, highlighting the broader implications of homotopy theory in areas such as algebraic geometry. By investigating how homotopy groups influence the topological structure of algebraic varieties and their associated schemes, the paper demonstrates the significant utility of homotopy theory in connecting abstract topological concepts with concrete algebraic structures. The challenges of working with exact sequences, particularly in the complex landscape of higher homotopy groups, are also addressed, underscoring the mathematical sophistication required to navigate these topics. This exploration provides valuable insights into the role of homotopy theory in modern mathematical analysis, emphasizing its deep and far-reaching impact across various fields.

**Keywords:** Algebraic Topology, Algebraic, Geometry.

## 1. Introduction

Research on homotopy groups is a central theme in algebraic topology, providing a deep understanding of the topological structure of spaces through the study of continuous deformations. Homotopy theory,

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which focuses on these deformations, is fundamental for analyzing how spaces can be continuously transformed into one another, and it forms the backbone of various mathematical fields such as algebraic topology and category theory [1]. The concept of homotopy groups, particularly the fundamental group, plays a crucial role in understanding the behavior of loops within topological spaces, offering insights into the properties of these spaces that are invariant under continuous transformations. Current research in homotopy theory has expanded to include applications in diverse mathematical domains, including algebraic geometry and category theory. Recent studies have demonstrated the utility of homotopy groups in understanding the topological structure of algebraic varieties and schemes, highlighting the importance of exact sequences and chain complexes in this context [2]. The application of homotopy theory to fibrations and loop spaces has also revealed profound connections between fundamental groups and the path-connected components of loop spaces, offering a new perspective on the topological invariants of simply connected spaces.

This paper builds upon the foundational concepts of homotopy groups, exploring their definitions and operations while constructing a chain complex and proving its exactness. The study applies homotopy theory to fibrations, deriving an exact sequence that elucidates the relationship between loop spaces and fundamental groups in simply connected spaces. Furthermore, the paper extends these theoretical results to the context of algebraic geometry, examining the implications of homotopy theory for algebraic varieties and schemes. The challenges associated with exact sequences and higher homotopy groups are also discussed, emphasizing the complexity and interdisciplinary nature of homotopy theory in modern mathematical analysis.

## 2. Relevant theories

### 2.1 Basic Concepts of Homotopy Groups

Denote $I$ as the unite closed interval, consider topological spaces $U$ and $V$, for any continuous maps $f$, g from $U$ to $V$, if there exists a continuous map $H: U\times I \to V$ such that H( $u$ ,0)=f( $u$ ) and H( $u$ ,1)=g( $u$ ) for any $u$ inside $U$ we say $f$ and g are homotopic. Homotopic is an equivalence relation, we say f is null homotopic if it is homotopic to a constant map [3]. For a subset A of $U$, if any u in A, f( $u$ )=H( $u$ ,t)=g( $u$ ) for any t in $I$ then $f$ and g are homotopic relative to A [4]. This relatively homotopic is also an equivalence relation. Consider a category, in this category, the objects are topological spaces, the morphisms are (relative) homotopic classes of maps so in this category we call the categorically isomorphic spaces homotopy equivalent [5]. Now in this category, let $(U,u_0)$ be a pointed topological space define $\pi_n(U,u_0)$ be the homotopic class of maps $(I^n,\partial I^n)\to(U,u_0)$ which is the homotopy class of maps $I^n\to U, \partial I\mapsto u_0$ for $n=0,1,2,\cdots$. For $n=0,1,2,\cdots$, $f(u_1,u_2\cdots,u_n), g(u_1,u_2\cdots,u_n)\in\pi_n(U,u_0)$, we define an operation [6].

$$f*g(u_1,u_2,\cdots,u_n)=\begin{cases} f(2u_1,u_2,\cdots,u_n) \text{if } 0\leqslant u_1\leqslant \frac{1}{2} \\ g(2u_1-1,u_2,\cdots,u_n) \text{if } \frac{1}{2}\leqslant u_1\leqslant 1. \end{cases} \quad (1)$$

This operation is well defined with respect to the homotopic class of $f$ and g which gives $\pi_n(U,u_0)$ a structure of group whose identity is the class of constant map $\bar{u}: I^n\to\{u_0\}$ [7]. Inverse of $f(u_1,u_2\cdots,u_n)$ is $f(1-u_1,u_2\cdots,u_n)$. And $\pi_0(U,u_0)$ is the path-connected components of $U$ so it is necessary to write $\pi_0(U)$ instead. We call $\pi_1(U,u_0)$ the fundamental group of $U$ at the base point $u_0$.

Consider a subspace $W$ of $U$, which contains $u_0$, we define $\pi_n(U,W,u_0)$ for each integer $n$ as a subgroup or subset of $\pi_n(U,u_0)$ consists of the homotopic class of maps $(I^n,\partial I^n,s_0)\to(U,W,u_0)$ which map the boundary of the unite cube into $W$ and some point of the boundary of the unite cube to $u_0$ [8].

The natural inclusion map $W\to U, u_0\mapsto u_0$ induces a group homomorphism $i:\pi_n(W,u_0)\to\pi_n(U,u_0), n=0,1,2,\cdots$. Inside the $\pi_n(U,u_0)$, $f:(I^n,\partial I^n)\to(U,u_0)$ is indeed a map $(I^n,\partial I^n,s_0)\to(U,u_0)$ so there is a natural homomorphism $j:\pi_n(U,u_0)\to\pi_n(U,W,u_0)$.

Upon the constant composition $(W,u_0)\to(U,u_0)\to(U,W,u_0)$, $j\circ i$ is the identity. And for any continuous map $f:(I^n,\partial I^n,s_0)\to(U,W,u_0)$, as well as, $(\partial I^n,s_0)$ is homotopy equivalent to $(I^{n-1},\partial I^{n-1})$, the restriction $f${\left.}\right|_$\partial I^n : (\partial I^n,s_0)\to(W,u_0)$

gives a homomorphism $\delta : \pi_n(U,W,u_0) \to \pi_{n-1}(W,u_0)$ and if $f$ maps the whole $\partial I^n$ to a point then its restriction is a constant map, as a result, $\delta \circ j$ is also the identity [9]. For any $f$ in the image of $i \circ \delta$, there is an extension of $f$ to $I^n \to U$, since $I^n$ that is simply connected, $f$ is null homotopic, hence $i \circ \delta$ is also the identity. As a result, the sequence

$$\cdots \to \pi_n(W,u_0) \overset{i}{\to} \pi_n(U,u_0) \overset{j}{\to} \pi_n(U,W,u_0) \overset{\delta}{\to} \pi_{n-1}(W,u_0)$$
$$\to \cdots \to \pi_1(U,W,u_0) \to \pi_0(W,u_0) \to \pi_0(U,u_0). \quad (2)$$

Is a chain complex, note that we define the kernels at the level of $\pi_0$ of the pointed space are the path components whose image is homotopic to the homotopic class of the base point we chose.

Claim: This Chain complex is exact [10].

Proof of Claim: If $f : (I^n, \partial I^n) \to (W,u_0) \overset{inclusion}{\to} (U,u_0)$ is null homotopic then this homotopy gives an extension $I^{n+1} \to X$ so $f$ is inside the image of $\delta$. For any null homotopic $f : I^n \to X$ inside $(U,W,u_0)$, the map is homotopic to the constant map $I^n \to \{u_0\}$ inside $(U,u_0)$ hence $f$ is in the image of $i$. Finally, for any $f : (I^n, \partial I^n, s_0) \to (U,W,u_0)$ whose restriction on the boundary $f\{\left.}\right|\_\partial I^n$ is null homotopic, this gives a homotopy between and a constant map and $(I^n, \partial I^n) \to (U,u_0)$ so $f$ is in the image of $j$.

### 2.2 Fibrations and the Applications of Homotopy

Continuous map $Fi : X \to Y$ is a fibration if for any topological space $A$, and commutative diagram

$$A ? X$$
$$\downarrow^{a \mapsto (a,0)} \downarrow Fi \quad (3)$$
$$A \times I ? Y$$

There is a morphism of topological spaces $A \times I \to X$ which makes the diagram commutative after adding this arrow to the diagram.

Then, by the universal property of fiber product, consider any cartesian diagram

$$X \times_Y S ? X$$
$$\downarrow Fis \downarrow Fi \quad (4)$$
$$S ? X$$

If $Fi$ is a fibration then so is $Fi_S$. So for the following cartesian diagram

$$F_{y0} ? X$$
$$\downarrow \downarrow Fi \quad (5)$$
$$\{y_0\} ? Y$$

Where $Fi : X \to Y$ is continuous, $F_{y_0} ? Fi^{-1}(y_0)$ if $Fi$ is a fibration then so is $Fi\{\left.}\right|\_F_{y_0}$. Assuming $Fi$ is a fibration, after choosing $x_0 \in F_{y_0}$, there exists a long exact sequence

$$\cdots \to \pi_n(F_{y_0}, x_0) \overset{i}{\to} \pi_n(X, x_0) \overset{j}{\to} \pi_n(Y, y_0) \overset{\delta}{\to} \pi_{n-1}(F_{y_0}, x_0)$$
$$\to \cdots \to \pi_1(Y, y_0) \to \pi_0(F_{y_0}, x_0) \to \pi_0(X, x_0) \quad (6)$$

Claim: $\pi_n(X, F_{y_0}, x_0) isisomorphicto \pi_n(Y, y_0)$ for each integer $n = 1, 2, \cdots$

Proof of Claim: Consider $f : (I^n, \partial I^n) \to (Y, y_0)$ and a commutative diagram:

$$I^{n-1} \overset{t \mapsto x_0}{\to} X$$
$$\downarrow_{t \mapsto (t,0)} \downarrow^{\downarrow i} \quad (7)$$
$$I^n \overset{f}{\to} Y$$

There exists $\bar{f} : I^n \to X$, since $f$ maps $\partial I^n$ to $y_0$, $\bar{f}$ maps $\partial I^n$ to $F_{y_0}$. Now for any $f : (I^n, \partial ? I^n) ? \to ? (X, F_{y_0})$ such that $Fi \circ f$ is null homotopic, say this homotopy is $H : (I^n, \partial ? I^n) \times I ? \to ? (Y, y_0)$, hen lifting $H$ to $\bar{H} : I^n \times I \to X$, which gives a homotopy between $f$ and a map of pairs whose image is inside the $F_{y_0}$ hence it is the identity in $\pi_n(X, F_{y_0}, x_0)$.

As a result, the following is exact:

$$\cdots \to \pi_n(F_{y_0}, x_0) \overset{i}{\to} \pi_n(X, x_0) \overset{j}{\to} \pi_n(Y, y_0) \overset{\delta}{\to} \pi_{n-1}(F_{y_0}, x_0)$$
$$\to \cdots \to \pi_1(Y, y_0) \to \pi_0(F_{y_0}, x_0) \to \pi_0(X, x_0) \quad (8)$$

## 3. Applications

### 3.1 The Application in Loop Space of a Simply Connected Topological Space

Consider space $C(I,Z)$ with respect to a simply connected topological space $Z$ consists of the continuous maps from $I$ to $Z$, equipped with compact-open topology. Then the continuous map $C(I,Z) \to Z \times Z, f \mapsto (f(0), f(1))$ is a fibration, indeed for any topological space $A$ and

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commutative diagram

$$A \overset{a\to fa}{\to} X$$
$$\downarrow\downarrow \quad (9)$$
$$A\times I ? Z\times Z$$

$A\times I \to C(I,Z),(a,t)\mapsto f_a((1-t)x)$ for $x\in[0,1]$ which gives a lift as we desired.

Consider $P_{x_0}Z \subseteq C(I,Z)$ the subspace consisted of the continuous maps who map 0 to $x_0$. Then following the Cartesian diagram

$$P_{x_0}Z ? C(I,X)$$
$$\downarrow\downarrow \quad (10)$$
$$\{x_0\}\times Z ? Z\times Z$$

This map $p: P_{x_0}Z \to Z, f\mapsto f(1)$ is a fibration. In the pointed topological space $(Z,x_0)$ the loop space at $x_0$ is $_{x_0}Z ? p^{-1}(x_0)$. From what we did before, from the exactness at $\pi_1(Z,x_0)$ and $\pi_0(_{x_0}Z)$ of $\pi_1(P_{x_0}Z)\to\pi_1(Z,x_0)\to\pi_0(_{x_0}Z)\to\pi_0(P_{x_0}Z)$, as well as $\pi_1(Z,x_0)=*$ and $\pi_0(P_{x_0}Z)=*$, we get that $\pi_1(Z,x_0)\to\pi_0(_{x_0}Z)$ is bijective. As a result, path component of loop space at $x_0$ "looks like" the fundamental group of $Z$ with the base point $x_0$.

### 3.2 Application in Algebraic Geometry

In algebraic geometry, homotopy theory provides powerful tools for studying the properties of algebraic varieties and schemes. By applying the concepts of homotopy groups, we can investigate the topological structure of algebraic varieties and their relationship with other algebraic objects. Specifically, the fundamental group of an algebraic variety offers insights into its covering spaces, which are closely related to the variety's geometric properties.

For instance, consider a smooth projective variety X over a field k. The fundamental group $pi_1(X)$ plays a crucial role in understanding the etale cohomology of X, which in turn provides information about the variety's arithmetic properties. Additionally, homotopy theory can be employed to examine the fibration structures in algebraic varieties. A fibration in the context of algebraic geometry corresponds to a morphism between varieties, and understanding its homotopy properties can help in analyzing the variety's fundamental group and its higher homotopy groups. Moreover, the application of homotopy theory to schemes extends the concept to the algebraic category, allowing for the exploration of more complex structures. By considering the homotopy classes of morphisms between schemes, one can study the homotopy type of a scheme, which reveals significant information about its algebraic and geometric structure.

## 4. Challenges

Complexity of Homotopy Theory: Understanding the deep and intricate relationships within homotopy groups and their applications in topological spaces requires a strong foundation in both topology and category theory. The abstraction and generalization involved can be challenging for those new to the field.

Exact Sequences in Homotopy: Proving the exactness of sequences in homotopy theory, particularly when dealing with fibrations and loop spaces, is a complex task. It involves intricate reasoning and careful handling of topological constructs, which can be difficult to follow and apply correctly.

Applications in Algebraic Geometry: Applying homotopy theory to algebraic geometry, particularly in the context of algebraic varieties and schemes, presents additional challenges. The interplay between abstract topological concepts and concrete algebraic structures requires a deep understanding of both domains, making the application of homotopy theory in this area particularly challenging.

Computation of Homotopy Groups: Computing higher homotopy groups is notoriously difficult, even for relatively simple topological spaces. The complexity increases significantly when dealing with more intricate spaces, and often, explicit computations are either infeasible or require sophisticated techniques.

Interdisciplinary Knowledge: Successfully navigating the challenges presented by homotopy theory and its applications requires interdisciplinary knowledge that spans algebraic topology, algebraic geometry, and category theory. This broad scope can be overwhelming and demands a high level of mathematical maturity and expertise.

## 5. Conclusion

This paper has provided a comprehensive exploration of homotopy groups, emphasizing their foundational role in understanding the structure of topological spaces. By constructing a chain complex and proving its exactness, the study elucidated the intricate relationship between loop spaces and fundamental groups in simply connected spaces, thereby offering new insights into the interpretation of fundamental groups through homotopy. Furthermore, the application of these theoretical results to algebraic varieties and schemes demonstrated the broader utility of homotopy theory in bridging abstract topological concepts with concrete algebraic structures. Looking forward, future research could focus on extending these concepts to more complex and less studied areas, such as higher-dimensional algebraic varieties and non-simply connected spaces. Additionally, further investigation into the computational aspects of higher homotopy groups could yield new methods for tackling the inherent challenges in this field. Exploring the interplay between homotopy theory and other mathematical domains, such as differential geometry and mathematical physics, may also reveal deeper connections and novel applications, further solidifying the importance of homotopy theory in modern mathematical analysis.

## References

[1] Torzewska Fiona. Topological quantum field theories and homotopy cobordisms. Applied Categorical Structures, 2024, 32(4): 22.

[2] Szegedy Balazs. A higher-order generalization of group theory. arXiv preprint, 2024, arXiv:2407.07815.

[3] Chowdhury Chirantan. Motivic homotopy theory of algebraic stacks. Annals of K-Theory, 2024, 9(1): 1-22.

[4] Krishnan Sanjeevi, Ogle Crichton. The uniform homotopy category. Journal of Pure and Applied Algebra, 2024, 228(1): 107425.

[5] Weinzierl Simon Maximilian. On the H_0^{A^1} of classifying spaces of algebraic groups. Diss. lmu, 2024.

[6] Zhu Xianxun, Huang Yao, Wang Xiangyang, Wang Rui. Emotion recognition based on brain-like multimodal hierarchical perception. Multimedia Tools and Applications, 2024, 83(18): 56039-56057.

[7] Myers David Jaz, Sati Hisham, Schreiber Urs. Topological Quantum Gates in Homotopy Type Theory. Communications in Mathematical Physics, 2024, 405(7): 1-114.

[8] Zakharevich Inna. K-theory and Characteristic Classes: A homotopical perspective. 2024.

[9] Gaucher Philippe. Natural homotopy of multipointed d-spaces. arXiv preprint, 2024, arXiv:2404.18693.

[10] Abouzaid Mohammed, Blumberg Andrew J. Foundation of Floer homotopy theory I: Flow categories. arXiv preprint, 2024, arXiv:2404. 03193.