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# Study of Random Walk: Basic Theory and Multiple Application

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**Abstract:**

Random walk is a randomly walk process that describes the path including a succession of random steps in the mathematical region. A procedure for estimating the likely position of a point exposed to random motions in probability theory, given the probabilities (which remain constant at each step) of traveling a certain distance in a certain direction. A Markov process that exhibits future behavior independent of prior history is the random walk. This paper will discover the classic recurrence Theorem concerning random walks on the d-dimensional integer lattice, and demonstrate the leads to complete generality with simple methods supported by String's Approximation. Furthermore, this essay will explore the situation that walker randomly walks in different dimensions and discover that random walk on dimensions one and two will return to the initial point, dimension three and higher dimensions will not. The meaning of this essay is to help more people have some basic understanding with random walk and to know there are some applications of random walk.

**Keywords:** String's Approximation, Recurrence Theorem, Random walk.

## 1. Introduction

A random walk, also referred to as a drunkard's walk, is a stochastic process used in mathematics to describe a path made up of a series of random steps on a mathematical space. A path made up of a succession of random steps is described mathematically as a random walk. It is frequently used to simulate random processes in a variety of domains, including computer science, biology, economics, and physics. In one dimension, the most basic type of random walk is one in which a walker goes one unit to the left or one unit to the right with equal probability at each step [1].

The random walk on the integer number line, which begins at 0 and advances +1 or −1 with equal chance at each step, is a simple example of a random walk. A molecule's passage through a liquid or gas (see Brownian motion), an animal in quest of food, the price of a volatile stock, or a gambler's financial situation are a few other examples [2]. Numerous scientific domains, such as ecology, psychology, computer science, physics, chemistry, biology, economics, and sociology, as well as engineering, employ random walks. Karl Pearson coined the phrase "random walk" in 1905. The concept can extend to multiple dimensions, including two-dimensional and three-dimensional spaces. In these cases, the walker can move in any direction, making the process more complex and applicable to more sophisticated systems [3].

Random walks have a close relationship with probability theory, particularly in the study of Markov chains and Brownian motion [4]. In fact, the continuous limit of a random walk, where the step size approaches zero, leads to Brownian motion, a fundamental model in stochastic processes. The study of random walks is not only important in theoretical mathematics but also in practical applications. For instance, in computer science, random walks are used in algorithms for network analysis and optimization. Overall, random walks offer a powerful tool for understanding and simulating complex systems driven by random forces

## 2. Method and Theory

This part of essay will show the simple random walk on dimension one and two is recurrent and the dimension three even higher dimensions is transient. Defined that $u_n$ =probability of return to the start positions at step n. $f_n$ =probability of return to start point for the first time. Then it is evident that p=$\sum_{i=1}^{\infty} f_i$. To find out the sum of $f_i$, consider that the relationship between the f&u, which u is relatively easier to calculate. $u_0=1, f_0=0$. Furthermore, $u_{0=}u_0, u_{1=} f_0u_1 + f_1u_0$ , $u_2 = f_0u_2 + f_1u_1 + f_2u_0$ , which continues to [5]

$$u_n = f_o(u_n) + f_1(u_{n-1}) + \ldots + f_n u_0 \tag{1}$$

Therefore, U(x)= $\sum_{i}^{\infty} u_i x^i$ . In the same way, F(x)= $\sum_{j=0}^{\infty} f_j x^j$ .What is more, based on the formula, U(x) F(x) = $u_0 f_0 + (u_0 f_1 + u_1 f_0)x + (u_0 f_2 + u_1 f_1 + u_2 f_0)x^2 + \ldots =$ $1 ※ 0 + u_1 x + u_2 x^2 + \ldots = U(x) - 1$, and than divide both side by U(x): F(x)=1- $\frac{1}{U(x)}$, apply x=1, if U(1)=1,which means it is recurrent, if U(1)< $\infty$, which means it is transient.

Additionally, for any odd steps, the probability of returning to the start position is 0. U(1)= $\sum_{k=0}^{\infty} u_k = \sum_{n=0}^{\infty} u_{2n}$ .Consequently, there is only calculation for the even steps in random walk. Moreover, the random walk in dimension one have two directions: go left or go right, the probability of each direction is basically 0.5 [6]. For the calculation that is to proving the random walk in dimension one is recurrent or transient is the formula which named Stirling's approximation: n!~ $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ , $u_{2n} = \frac{\binom{2n}{n}}{2^{2n}}$ , and [7]

$$\binom{2n}{2} = \frac{(2n)!}{n!n!} \backsimeq \frac{\sqrt{2\pi 2n}(\frac{2n}{e})^{2n}}{(\sqrt{2\pi n}(\frac{n}{e})^n)^2} \tag{2}$$
$$= \frac{\sqrt{2} \times \sqrt{2\pi n} \times (2)^{2n} \times (n)^{2n} \times e^{-2n}}{(\sqrt{2\pi n})^2 (n)^{2n} \times e^{-2n}} = \frac{2^{2n}}{\sqrt{\pi n}}$$

Thus, $u_{2n} = \frac{\binom{2n}{n}}{2^{2n}} \cong \frac{\frac{2^{2n}}{\sqrt{\pi n}}}{2^{2n}} = \frac{1}{\sqrt{\pi n}}$ and $U(1) = \sum_{n=0}^{\infty} u_{2n} \cong \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{n^{1/2}} = \infty$ . So for dimension one, P( $s_{2n} = 0) =$ $F(1) = 1 - \frac{1}{\infty} = 1$. Therefore, in dimension one of random walk is this case is mathematically proving that is recurrent, what is more, in the Stirling approximation, after the calculation, the root must be looking like $\sum_{1}^{\infty} \frac{1}{n^p}$ . For the judgement that is to know it is divergent (recurrent) or convergent (transient) is to see the amount of p. If p$\leqslant$1 , which means it is divergent, if p>1, it is convergent, in dimension one, p is 0.5, so random walk on dimension 1 is recurrent. In fact, random walk on dimension one also can seen as a Markov chain.

In the two dimensions case, the directions are increase from two to four, which is go left, go right, go up or go down. In this case, the probability of each direction is basically 1/4. So, $u_{2n} = (\frac{1}{4})^{2n}$.The number of return paths is $\frac{2n!}{x!x!(n-x)!(n-x)!}$ , in this formula, the x means the any direction in right and left or up and down, so n-x means that the opposite direction of x. The total number of return paths is that $\sum_{x=0}^{n} \frac{2n!}{x!x!(n-x)!(n-x)!}$ . Plus, the probability of 2n-th step return is the probability of $u_{2n}$ times the total number of return paths, which is $\left(\frac{1}{4}\right)^{2n} \sum_{x=0}^{n} \frac{2n!}{x!x!(n-x)!(n-x)!}$. This solution is approxi-

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mately equals to $\frac{1}{n^1}$, the number of p in this essay mentions in dimension 1 is equals to 1, which means the random walk in dimension 2 is divergent or recurrent.

The formula that to prove it is divergent is binomial series: $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$ , where $(\,^n_r) = C^n_r = \frac{n!}{r!(n-r)!}$. Moreover, to mathematically proving it is recurrent , $(1+x)^{2n}$ can be written be $(1+x)^n(1+x)^n$, and use the binomial series to extended this function, it will equals to $(C^o_n 1^0 x^n + C^1_n 1^1 x^{n-1} + \ldots C^{n-1}_n 1^{n-1}x^1 + C^n_n 1^n x^0)^2$ , and it can combined the coefficient of $x^n$, combined one or more sequences before and after each sequences, each sequences one can have $x^n$, and organize the coefficient of $x^n$ ,which will equals to $(C^{0^2}_n + C^{1^2}_n + \ldots C^{n^2}_n)x^n = C^n_{2n}1^n x^n = C^n_{2n}x^n$. So, $\sum_{x=0}^{n}(\,^n_x)^2 = (\,^{2n}_n)$ .Let's turn to dimension three, obviously dimension three has six directions, which means the probability of each direction is $\frac{1}{6}$. So, in this case, $u_{2n} = (\frac{1}{6})^{2n}$. If $k(k \geqslant 0)$ or $j(j \geqslant 0)$ is show the any one direction, j + k $\leqslant n$, 2k+2j+2(n-k-j)=2n.

The number of walks in dimension three that return to the start position in 2n steps:

$$\sum \frac{2n!}{k!k!j!j!(n-k-j)!(n-k-j)!} \quad (0 \leqslant k, j), (k+j \leqslant n)$$

. Because there is a bijection between the amount of rearrangements of j A's, k C's, and n-j-k L's. and the number of n letter string of A's, C's or L's. there are $3^n$ such n letter strings, from which it concludes the observation. Now it is possible t show that $\sum \frac{1}{3^{2n}} \frac{2n!}{k!k!j!j!(n-k-j)!(n-k-j)!}$

$$\leqslant max(3^{-n}\frac{n!}{k!j!(n-k-j)!}) \quad (0 \leqslant k, j), (k+j \leqslant n),$$

from which it follows that

$$u_{2n} \leqslant 2^{-2n}\binom{2n}{n} max(3^{-n}\frac{n!}{k!j!(n-k-j)!}).$$

In this case, using the formula Stirling's approximation again, the function will transferred be that:

$$\frac{n!}{k!j!(n-k-j)!} \frac{\sqrt{2\pi n}(\frac{n}{e})^n}{\sqrt{2\pi j}(\frac{j}{e})^j\sqrt{2\pi k}(\frac{k}{e})^k\sqrt{2\pi(n-j-k)}(\frac{n-j-k}{e})^{n-j-k}}.$$

It is the initially exercise in analysis to look that j=k=n-j-k $\approx \frac{n}{3}$ maximizes this value. For ease for calculation, suppose that $\frac{n}{3}$ is integer, and get that

$$max(3^{-n}\frac{n!}{k!j!(n-k-j)!})$$

$$= (3^{-n})\left(\frac{\sqrt{n}}{\sqrt{\frac{n}{3}\times\frac{n}{3}\times\frac{n}{3}}}\right)\left(\frac{n^n}{(\frac{n}{3})^{\frac{n}{3}}(\frac{n}{3})^{\frac{n}{3}}(\frac{n}{3})^{\frac{n}{3}}}\right)$$

$$= (3^{-n})\left(\frac{c\sqrt{n}}{n\sqrt{n}}\right)\left(\frac{n^n}{(\frac{n}{3})^n}\right) = (3^{-n})\left(\frac{c}{3}\right)(3^n) = \frac{c}{n}.$$

Where c is probably equal to $3\sqrt{3}$, but for this paper's purpose is simply some constant. Combining all of this, can see that

$$u_{2n} \leqslant 2^{-2n}\binom{2n}{n}\left(\frac{c}{n}\right)\frac{1}{\sqrt{\pi n}} \times \frac{c}{n} = \frac{c}{n^{\frac{3}{2}}} \to \sum_{n=o}^{\infty}u_{2n} = \sum_{n=0}^{\infty}\frac{c}{n^{\frac{3}{2}}} < \infty$$

,and based on the theory mentions in dimension one, random walk in dimension three is transient, which is convergent. Observe that a tuple of at least three integer coordinates can be used to represent any random walk in dimension four or higher. But since this article has already shown that random walks in dimension three are convergent, the initial three coordinates will never simultaneously return to 0 in any higher dimension that this paper randomly travels in. Consequently, random walks in any higher dimension are convergent.

## 3. Application of Random Walk

Random walks are fundamental in modeling sequences of random events, such as a particle moving in discrete steps on a number line. This approach provides insight into critical probabilistic concepts like expected values, variances, and limit distributions, which are central to solving problems in diverse areas like finance (e.g., modeling stock prices) and statistical physics (e.g., particle diffusion).

Beyond mathematics, random walk serves as vital tools in other disciplines. In physics, for example, they are used to model the diffusion of particles. A continuous-time random walk can describe the erratic movement of particles suspended in a fluid, which is fundamental to understanding heat transfer, molecular diffusion, and other key processes in material science.

Stochastic Processes is a key example of a stochastic process, random walks describe systems that evolve randomly over time. One notable application is in modeling Brownian motion, where the random movement of particles in a fluid is studied to understand diffusion processes. This framework is vital in fields like thermodynamics, material science, and chemical kinetics, where understanding how substances mix or spread is crucial.

Additionally, random walks underpin the theory of Markov chains, where the transition to the next state depends only on the current state. Markov chains are widely applied to model diverse phenomena, from financial markets to genetics. In mathematics, they help analyze long-term behavior and steady-state distributions, providing critical insights into complex dynamic systems.

Moreover, in graph theory, random walks are used to study the properties of complex networks, such as social networks, transportation systems, or the internet. By modeling a random walk on a graph, one can identify key nodes, measure network connectivity, and understand information propagation. The PageRank algorithm, which forms the backbone of search engine optimization, uses random walks to rank web pages by importance.

What is more, random walks are instrumental in harmonic analysis, particularly in studying functions on groups or other algebraic structures. For instance, random walks on a lattice can approximate solutions to the Dirichlet problem in potential theory, helping to understand phenomena like electrostatics and fluid dynamics. This application bridges probability theory and functional analysis, providing tools to solve partial differential equations.

In ergodic theory, random walks help analyze the statistical properties of dynamical systems over time. By examining the long-term behavior of a random walk, mathematicians can study how a system “mixes” or reaches equilibrium. This has applications in understanding chaos and randomness in various mathematical contexts, such as number theory and statistical mechanics.

In economics, people use random walks model to simulate the unpredictable behavior of stock prices, consistent with the Efficient Market Hypothesis (EMH), which posits that price changes are random and reflect all available information. This randomness is also fundamental in the Black-Scholes model for options pricing, which assumes that asset prices follow a geometric random walk to derive fair value estimates.

In biology, random walks describe molecular diffusion, such as the movement of nutrients, proteins, or drugs within cells. They also model animal foraging patterns, capturing how animals optimize their search for food in complex, stochastic environments. This approach provides insights into ecological dynamics, evolutionary biology, and the behavior of biological systems at multiple scales.

Above all the text this paper mentions are that related with theory or any aspect on academic. Let people now turn to actual application in their daily life. In the recent years, TikTok has been a most popular App in China even around the world. Recommender system has considerable contribution for it, this technology has significant effect to personalize each individual account, find the preferences of every people who use this app and satisfy the demand, which will appeal consumers keep use this app even suggest to people who around him, this phenomenon is not just on TikTok, Twitter and Instagram also contribute to this system. So, what is recommender system, how it works?

Recommender systems are an innovative system which help consumers to find preference goods. In this paper, it will present three ways to create a list recommendation. The first method this paper present is Item Rank, a random walk according to scoring algorithm, which can use to rank goods based on the prediction of user’s preferences to suggest top-rank items to potentially appeal consumers. The second technique for resolving the paper recommendation problem is the Paper Rank, also known as the Page Rank algorithm, which is based on random walks. It has an Item Rank-like structure. They locate relevant publications for experiment study areas by using the model represented by the citation graph.

Overall, random walks serve as a unifying framework for analyzing randomness and uncertainty across a wide range of contexts. Whether modeling stock prices, simulating molecular movement, exploring network dynamics, or optimizing algorithms, random walks capture the core of systems governed by chance. Their broad applicability and fundamental nature underscore their importance in both theoretical research and practical problem-solving, making them indispensable tools in modern science and mathematics.

## 4. Conclusion

To sum up, on the one hand, this essay has introduce the background and basic information related with random walk. What is more, this paper shows the academic theory with random walk and explain the prove process which demonstrate the characteristic about random walk in dimension one, dimension two and dimension three even higher dimensions. In this case, this essay shows the theory that random walk in dimension one is recurrent, which means the particle or something walk randomly in dimension one will return to the start point. The situation

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on random walk in dimension two is same with dimension one. In contrast, the situation on random walk in dimension three is different with dimension one and dimension two, in dimension three, random walk in transient, which means the walker will not return to the start point, the random walk in higher dimensions also is transient. In this text, Stirling's approximation and binomial series is the mathematical theory that this paper used to prove the random walk is transient or recurrent. In the period of time that study with random walk, actually there are many of applications of random walk in people's daily life. For instance, exclude the larger proportions of application in academic, this essay has a specific example that is app named Tiktok. In fact, there are not only this app use the recommendation system which is the example of application in random walk. On the other hand, this essay has lots of development, such as the more specific explain and discover in this paper. In the days to come, the author wish develops the understanding on random walk, mathematics, and statistics in university.

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