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The Residue Theorem and Its Generalization

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Abstract:

In multivariate analysis, the residue theorem is an effective tool to calculate the results of analytic functions along closed curves. It can also be used to compute the integrals of real functions. The residue theorem reveals the relationship between integrals and singular residue on closed paths. If the Laurent expansion for a given point is computed, the corresponding residue for calculating the circumference integral can directly be obtained. It's a generalization of Cauchy's integral theorem and Cauchy's integral formula. In this paper, the definition of the Residue will be mentioned. Then the proof of Residue Theorem will be shown. Three examples of how to use the Residue Theorem to the closed curve integral takes the derivative. Finally, the generalization of residue theorem is mentioned. In mathematical analysis and practical life problems, some primitive functions that cannot be represented by elementary functions as integrands can be calculated using the residue theorem. Converting integrals into residues for calculation can make integral problems simpler.

Keywords: Residue theorem; integral; generalization.

1. Introduction

In complex analysis, the remainder theorem is a powerful tool for calculating the path integral of an analytic function along a closed curve, and can also be used to solve the real function integral. It is a generalization of the Cauchy's integral theorem and the Cauchy's integral formula. It occurs when calculating the eigenfunction of the Cauchy distribution, which is impossible to compute with the original calculus. This integral is expressed as the limit of a path integral that follows the real line from $-a$ to a, and then counterclockwise from a to $-a$ along a semicircle centered on 0, taking a greater than 1 so that the imaginary unit i is enclosed within the curve.

In 2003, Zhang discussed the intrinsic relation between residue theorem and integral of complex variable function. The close relation between residue theorem and Cauchy theorem, Cauchy formula and higher derivative formula is illustrated with examples [1]. In 2010, Zhang applied the residue theorem. The problem of calculating integral is transformed into calculating residue by using the structure integrand function [2]. In 2009, Zhang derived the K-residue theorem and its application in real integrals on the basis of K-residue. The conclusion obtained is the continuation and application of corresponding results in analytic functions and conjugate analytic functions [3]. In 2011, Wang proved summation formula of generalized trigonometric functions with parameters by using the corselius number theorem to calculate the calcium channel integral [4]. In 2012, Xu and Zhang introduced a new method for calculating definite integrals. By applying the residue theorem, the calculation of several types of real integrals is transformed into the calculation of complex integrals. This thesis is helpful to extend ideas for calculating definite integrals. [5]. In 2014, Zhi and Li introduced the residue theorem, a very useful theorem in complex variable function. Then, the residue theorem is applied to transform several real function integrals into complex function integrals [6]. In 2016, Ma discussed relations between residual theorems or residual theorems of infinite points in complex functions and Cauchy theorem, Cauchy formula, generalization of Cauchy theorem, integral expression of derivatives and Cauchy formula of infinite points [7]. In 2006, Shen and Jin illustrated the methods of calculating some generalized integrals and special definite integrals by residue theorem [8]. In 2020, Shen, Shao, Sun, et al solved the problem that the mean square error between the calculation result and the theoretical value of the traditional electromagnetic field numerical calculation system, a numerical integration calculation system based on the residue theorem is designed [9]. In 2020, Qiu wrote the integral value of a complex function at infinity approaches the integral value of another complex function. By applying the residue theorem, a class of analytic expressions for the infinite summation of all the roots of transcendental equations is obtained. The correctness of the analytical expression is verified by numerical calculation [10].

2. Residue Theorem

Before talking about the Residue Theorem, the definition of the Residue will be mentioned. It refers to the integral value of the analytic function along a simple forward closed curve. If the function f(x) has an isolated singularity close to a point A, then the integral value $(\frac{1}{2\pi i})$ $\int |x-A|$=R(x) dx is called the residue of f(x) with respect to the point A, denoted as Res $\left[f(x), A\right]$.

If there is a function called F and there is a is isolated singular point Z_0 of the function F. There will be positive number R_1 such that F is analytic at each point Z for which 0<$|Z-Z_0|<R_2$. Therefore $F_{(z)}$ has a Laurent series representation,

$$F(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \dots \tag{1}$$

where the coefficients a_n and b_n have certain integral representations.

In particular, $b_n = \frac{1}{2\pi i}\int_c f(z)dz$ or $\int_c f(z)dz = 2\pi i b_1$.

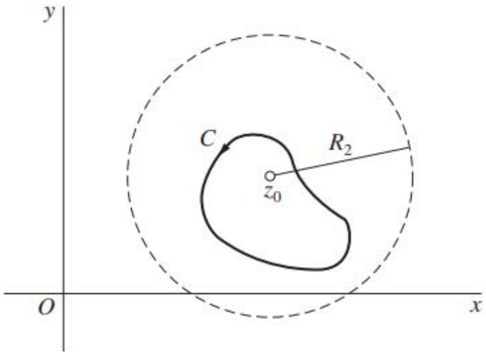

Fig.1 Picture One

In Fig.1, the complex number b_1, which is the coefficient of $\frac{1}{(z-z_0)}$ in expansion (a), is called the residue of F at the isolated singular point z_0, and we usually write as

$$b_1 = Res_{z=z_0} \text{f(z)} \tag{2}$$

Then $\int_c f(z)dz = 2\pi i b_1$ is becomes to

$\int_c f(z)dz = 2\pi i Res_{z=z_0} f(z)$.

Sometimes simply use B to denote the residue when the function F and the point z_0 are clearly contours.

Example 1:

Compute the integral $\int_c \frac{e^z-1}{z^4}dz.$

To determine that residue, we recall the Maclaurin series representation

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}(|z|<\infty) \tag{3}$$

Use it to write

$$\frac{e^z-1}{z^5} = \frac{1}{z^5}\sum_{n=1}^{\infty}\frac{z^n}{n!} = \sum_{n=1}^{\infty}\frac{z^{n-5}}{n!}\left(0<|z|<\infty\right) \tag{4}$$

Therefore, $Res_{z=0}\frac{e^z-1}{z^5} = \frac{1}{4!} = \frac{1}{24}$.

So that $\int_c \frac{e^z-1}{z^4}dz = 2\pi i\left(\frac{1}{24}\right) = \frac{\pi i}{12}$.

Example 2:

Define $\text{f(t)} = \frac{az^3+bz^2+cz+d}{z^4-1}$ with a=6, b=i+1, c=16,

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d=1-i.

Evaluate the integrals $\int_{\sigma} f(t)dz$

Where $\sigma(t) = i + \frac{e^{it}}{2}$ $(0 \leqslant t \leqslant 2\pi)$;

Use the method of integration to simplify this equation get

$$\frac{az^3 + bz^2 + cz + d}{z^4 - 1} = \frac{A}{Z-1} + \frac{B}{Z+1} + \frac{C}{Z-i} + \frac{D}{Z+i} \quad (5)$$

Continue to abbreviate the equation and get:

W($z^3 + z^2 + z - i$)+X($z^3 + z^2 i - z - i$)+Y($z^3 - z^2 + z - 1$)+ Z($z^3 - z^2 i - z + i$) (6)

Since a=6, b=i+1, c=16, d=1-I, we can get

$$(6 = W + X + Y + Z) \quad (7)$$

$$(i + 1 = W + Xi - Y - Zi) \quad (8)$$

$$(16 = W - X + Y - Z) \quad (9)$$

$$(1 - i = W - Xi - Y + Zi) \quad (10)$$

Therefore, W=6, X=5, Y=-2, Z=-3 and get $\int \frac{6}{Z-1} + \frac{5}{Z+1} + \frac{-2}{Z-i} + \frac{-3}{Z-i} dz$.

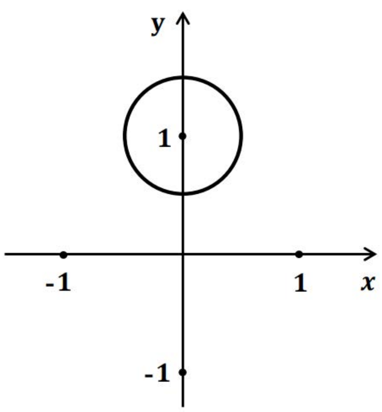

Fig.2 Picture Two

In the Fig.2, the radius of the function is $\frac{1}{2}$ since the $\frac{e^{it}}{2}$.

If $\int f(z)dz = 0$ than the point is differentiable on the closed curve.

So, there are four point which is (1,0), (-1,0), (0,1), (0,-1) and there is only one point in this equation. $\int f(z)dz = 2\pi i \times (-2) = -4\pi$.

Next, a generalization of the Residue Theorem will be given.

The Residue Theorem is a result obtained by people on the basis of studying the theorem of integration and series.

$$\int_c f(z)dz = 2\pi i Res_{z=z_0} f(z) \quad (11)$$

If C is a simple closed path in the forward direction, it resolves over the entire complex plane except for a finite number of points inside C. On this basis, people further study the application of the single Residue Theorem

$$\int_c f(z)dz = 2\pi i Res_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] \quad (12)$$

However, through observing the (12) function, the variables in the function f in the right have the following relation to the factors preceding f, a power is equal to the power of its variable plus 1. Therefore, under the same conditions, it can be extended to the following form.

If C is a simple closed path in the forward direction, it resolves over the entire complex plane except for a finite number of points inside C

$$\int_c f(z)dz = 2\pi i Res_{z=0}\left[\frac{1}{z^{k+1}} f\left(\frac{1}{z^k}\right)\right](k = 0,1,2,3\ldots) \quad (13)$$

Before we affirm this equation, we need to use some examples to improve it.

Find the following function $\frac{1}{z}$.

The positive integral of |z| = 1 along the circumference. If we use the (11) equation to calculate and get

$$\int_c \frac{1}{z} dz = 2\pi i Res_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

$$= 2\pi i Res_{z=0}(z \times \frac{1}{z^2})$$

$$= 2\pi i Res_{z=0}\left(\frac{1}{z}\right)$$

$$= 2\pi i \quad (14)$$

Use the same method when $k \geqslant 3$, the following equation can also be established

$$\int_c f(z)dz = 2\pi i Res_{z=0}\left[\frac{1}{z^{k+1}} f\left(\frac{1}{z^k}\right)\right](k = 0,1,2,3\ldots) \quad (15)$$

3. Conclusion

Residue theorem is an important theorem in complex analysis. It provides a method to calculate the integration of an analytical function along the trajectory of a closed curve. It's the promotion of Cauchy points Cauchy's theorem and integral formula and is widely used in engineering and mathematics. This calculation method is not only suitable for complex variable functions, but also can be applied to the integration of real functions, especially when dealing with some special types of real integrals, which can greatly simplify the calculation process. The application of the residue theorem is not limited to theoretical mathematics, it also plays an important role in solving practical problems. For example, when dealing with definite integrals and cosine functions, the residue theorem provides an efficient calculation, especially when dealing with sine and cosine functions. In quantum mechanics, the residue theorem is always be used to calculate the scattering amplitude and particle propagators, which helps to understand the interaction between particles and potential fields and predict the scattering behavior of particles. In the future, it hopes that the residue theorem can be applied to more fields to promote social development.

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