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Theory of Brownian Motion and its Applications in Selected Examples

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Abstract:

Random walk is the basic concept in mathematics. Brownian motion is one of the most popular application. The core of the Brownian motion is the continuous path for small particles. After being developed by Wiener, it is used in many other studies, including biology, chemistry, and economics. Brownian motion is contemporarily used for predicting possible path. For animals scientists can know the probable range of activities for animals in order to keep people from bothering them and keep them out of danger. For chemical reactions, it is easy to predict every tiny particle about their movement so that people can directly understand the core of the reaction and one can predict the speed and the phenomenon of the reaction. For economics, economists can use the Brownian motion to calculate the probability for stock to up or down. This paper aims to present the theory of Brownian motion and illustrate the applications. This allows people to better choose the time to buy and sell in order to get the highest profit, and makes the economic market more controllable.

Keywords: Brownian motion; Brownian Bridge Movement Model; Optical tweezers; Brownian Thermal Ratchet.

1. Introduction

For the random walk, there are five properties. The first one is randomness. The core of a random walk is that the move at each step is random and unaffected by the previous steps. This randomness shows irregularity. The second one is memoryless. Each step of the random walk is independent. This property suggests that the future state of a random walk is only related to the current state, not to past states. The third one is central limit. Under appropriate conditions such as step size independent distribution and finite variance, the position distribution of random walks will tend to be normal as the number of steps increases. The fourth one is diffusivity. The trajectory of a random walk will spread over a larger region of space over time. This diffusion is present in different dimensions of space, but the speed and manner of diffusion may vary from dimension to dimension. Here is a question: if there is a drunk man and a drunk bird, which one is more likely to go back home? The walk of a drunk man can be imagined as a random walk in two dimensions, but for the bird, it is a random walk in three dimensions. Therefore, the

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probability of people reaching their destination is larger than that of birds. The fifth one is the probability of returning to the original place in infinite times. By using p series, scientists conclude that in one or two dimensions, a random walk will certainly return to the original place in infinite times. But in three or more than three dimensions, a random walk will never return to the original place.

Brownian motion is a basic property in mathematics, but it has also been used in other studies. Through Brownian motion, many scientists have discovered many new technologies. They can improve people's living, broaden people's horizon and make the world better and better. In this paper, the author discusses the theory of Brownian motion, including its history, properties, and functions in section 2. In section 3, the author focuses on the application of the Brownian motion. The author lists the applications in biology, chemistry, and economics and imports many models, such as Brownian Thermal Ratchet.

2. Theory of Brownian Motion

2.1 History of Brownian motion

In 19 century, Brown observed through a microscope to discover that tiny pollen grains swarming in water in random motion. This is the first version of the Brownian motion. Conceptionally, Brownian motion is also recognized as the Brownian movement in some context. For clarification, the author will use Brownian motion throughout the paper. After that, in 1900, the Brownian movement was developed by Bachelier and independently by Einstein. And Einstein used Brownian motion to calculate Avogadro's number. In 1877, Delsaulx claimed that Brownian motion is caused by the imbalanced collision of small particles with surrounding liquid molecules.

After that, in 1900, the Brownian motion was developed by Bachelier and independently by Einstein. And Einstein used Brownian motion to calculate Avogadro's number. Beginning in 1920, Wiener proved a continuous path for the Brownian motion and published a series of papers devoted to the mathematical analysis of Brownian motion. They are: "mean of any metafunctional", "mean of analytic functional", "mean of analytic functional and Brownian motion", "differential space", and "mean of functional". Wiener concluded that the mean of the squared displacement of a particle in a liquid is proportional to time. Wiener hypothesized that the incremental momentum of a particle is obtained through the superposition of random collisions caused by the motion of the molecules of the liquid and the friction caused by the viscosity of the liquid. Wiener put the Brownian theory of motion on a firm footing by defining the mean as a Daniel integral. Although he studied Brownian motion in the paper, his method became a model for the modern theory of stochastic processes.

At first, mathematicians were not interested in Brownian motion, believing that it did not have much use for other studies. In 1964, Carker said that in the last twenty years, mathematicians have been engaged in a fruitful study of statistical processes, that is, the probabilistic analysis of phenomena that occur continuously over time. Statistical processes come from many fields of science, including astronomy, physics, genetics, economics, ecology, etc. The most famous example of a statistical process is the Brownian motion of a particle. In 1921, Wiener put forward the idea of Brownian motion theory based on measure theory on gas path set. It has been shown that this idea is very fruitful for probability theory. Not only does it breathe new life into old problems. More importantly, it opens entirely new research areas and suggests compelling connections between probability theory and other branches of mathematics [1].

2.2 Properties and Functions of Brownian Motion

Brownian movement has its definition in mathematic way. Let people make Ω, F, P be the probability space. A random process $(B(t,\omega), t \geqslant 0, \omega \in \Omega)$ is a Brownian motion if 1) For each fixed t, the stochastic variable $Bt = B(t,\cdot)$ has Gaussian distribution with mean value and variance t; 2) The process B increases stationarily and independently, and 3) For each fixed $\omega \in \Omega$, the path $t \to B(t,\omega)$ is continuous [2].

Brownian movement is one of the applications of random walk in three dimensions. Brownian movement is the random motion of tiny particles like pollen. Brownian movement is continuous random movement in the normal distribution. Brownian movement need to satisfy three conditions, which are $X(0)=0$, $X(t): t>0$ is right increasingly continuous, and for every s and t, $0 \leqslant s < t$, one can get

$$X(t) - X(s) \cong N\left(0, \sigma^2 (t-s)\right). \tag{1}$$

In addition, the Brownian movement is an example of the Markov chain. Therefore, Brownian movement have Markov property. For every $\tau > 0, B(t+\tau) - B(t): t > 0$. Brownian movement also has similarity property, i.e., for every constant $c \neq 0$, $\frac{1}{c} B\left(c^2 t\right): t > 0$. Through Brownian movement, one can get mean value function, variance function, auto covariance function, and auto correlation function.

Brownian movement is a special Markov chain. Therefore, Brownian movement has the property of Markov chain. For X in measurable state space (E, ϵ), $X=(\Omega, F,(Ft)t \geqslant 0,(Xt)t \geqslant 0,(Pt)t \geqslant 0,\{Px\}x \in E)$. For $P^{x} on(\mathcal{F}, \Omega)$, and $X(t)$ related to $\mathcal{F}(t)$, then

$$E_{x}[f(Xs+t) \mid Fs]=(Ptf)(Xs) \quad (2)$$

For $x \in E, A \in \epsilon$, $Ps+t(x,?A)=?\int_{E} P_{s}(x,dy) P_{t}(y,A)$. For $f(X_{s+t})=\mathcal{F}_{s}$, $(Ptf)(x)?=\int_{z} Pt(x,?dy) f(y)$. Each Brownian movement has its mean value, variance, autocovariance and autocorrelation [3]. The mean value function is $\mu_{B}(t)=E(B(t))=0$, the variance function is $\sigma_{B}^{2}(t)=Var(B(t))$, the autocovariance function is $C_{B}(s,t)=min\{s,t\}=s \wedge t$, and the autocorrelation function $r_{B}(s,t)=min\{s,t\}=s \wedge t$.

3. Applications

3.1 Brownian Motion in Biology

The Brownian Bridge Movement Model is one of the applications for Brownian motion. The authors, Inês Silva1 et al, provided dynamic Brownian Bridge Movement Models (dBBMMs) to receive radio-telemetry data from Ophiophagus Hannah, which is a species of snake,. In that experiment, the author uses two snakes. One is a juvenile male, and the other is an adult male. In the traditional skill home range estimation method, the author will have the general active area for each snake. However, in this method, both minimum convex polygons (MCP) and Kernel density estimators (KDE) existed at relatively high levels of Type I and Type II errors, which implied the inaccuracy of this species' activity space. Another method, dBBMMs, shows the accurate motion trail. It is more efficient to detect movement corridors and accurately represent the shelter's site in the long term. Brownian Bridge Movement Models can show every movement of each snake so that people can predict the further movement of snakes, which is really significant for conserving wild animals. People can protect and manage wild animals in places where they may appear so that wild animals are not affected by natural enemies and the ecological environment, so as to protect the diversity of the ecological environment [4].

Brownian movement can show animal movement. After a long time, one can get the probable range of activity for each animal. It is necessary for people to study marine animals. The author, Nicolas Humphries et al, shows that people use Brownian movement theory for the five largest preys in the marine (Thunnus obesus, Thunnus, albacares, Cetorhinus maximus). People can get vertical distribution for giant animals. Then, people can get the result of how the greenhouse effect influences huge animals [5]. By using the Brownian movement, it shows that after temperate increases, each shark will leave its habitat, where it has the best productivity. Sharks will dive deeper into marine life, which affects marine ecological balance.

Brownian motion has also contributed greatly to the study of single-celled organisms. In 1950, some biologists recognized that there are tremendous single-celled organisms in micrometer doing Brownian movement. For instance, Escherichia coli, a common bacterium that exists in people bodies, always does diffusion. After the experiment, the biologist showed that such diffusion was not diffusive. Every bacteria is doing random Brownian motion. In 1970, with highly development of technology, microscope technique has been greatly improved. Through a microscope, scientists have confirmed this idea. They concluded that there was a force on the bacteria, which could be measured.

In the 1940s, scientists discovered the Brownian Thermal Ratchet, which is a model used in proteins. As for humans, they will hydrolyze numerous proteins to convert ATP to ADP to creating mechanical work. In this process, there are some specific proteins that speed up reactions as enzymes. These proteins move along a trail of actin filaments. To discover how such proteins move, biologists discovered the Brownian Thermal Ratchet. It shows that particles diffuse and they receive an asymmetric space periodic force [6], see Fig. 1.

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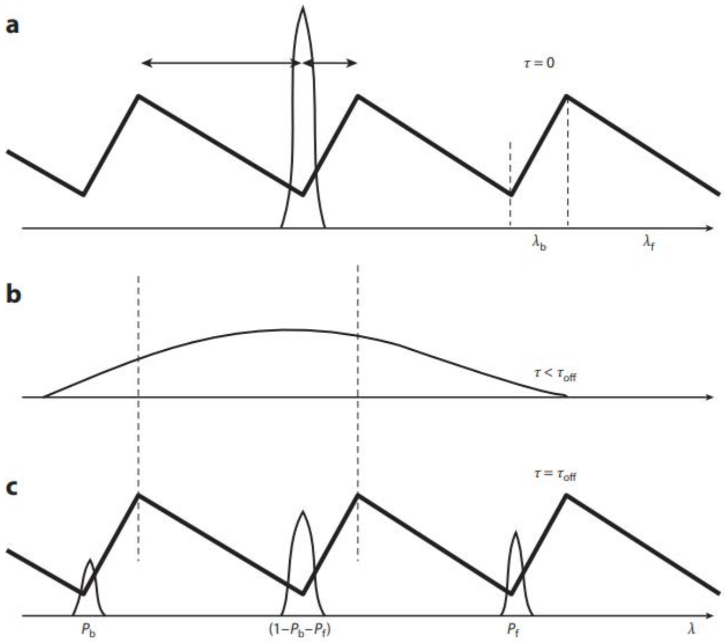

Fig. 1 Illustration of Asymmetric Potential and Particle Probability Density [6].

After that, scientists conclude that with lack of external force, tiny particles will not make any macroscopic excursion. Similarly, the outside periodic forces, whether they are asymmetric or not, will not contribute to obvious movement. That is because there is no related large-scale gradient force or the force is counteracted. Nevertheless, when the potential is asymmetric, the time modulation between random diffusion and inflicted potential will induce distinct drift of Brownian particles.

For the drift of Brownian particles, the Fig. 1 includes a principle that the field traps the particles in potential minima. People can discover that particles will make randomly diffusion when the field turns off. After the modulation time, when the field is switched on, particles will follow the gradient of the potential, and they will be in the minimum value. Because of the asymmetry of the potential, it represents more time for average diffusion on left-hand barrier than right-hand.

3.2 Brownian Motion in Chemistry

In chemistry, the Brownian movement is also significant. In the solution, there are tremendous tiny particles moving irregularly. They are all engaged in irregular movement, and they will react when particles collide together. Therefore, if people can see the motion of every particle in the solution, one can know the speed and phenomenon of the reaction. Carlos J. Bustamante et al. explain how optical tweezers utilize Brownian motion to manipulate particles in solution. After building the Brownian motion model, people can directly see the motion of particles under every condition, like temperature, pressure, acid, and so on. people cannot show every particle's movement, so people can follow a group of articles to contain the asynchronous contribution of all particles to get the average number. The average number can show the speed of the reaction. Because every particle is doing Brownian movement, chemists can use optical tweezers to see the reactions, especially for single molecules [7]. Optical tweezers is a technique to inflict forces or torques on a specific particle, which can clearly calculate the forces or torques created in the chemical reactions. Through optical tweezers, people can manipulate each particle by using photons with high energy. After people can control the particles, they can manipulate the reaction rate to avoid energy waste.

3.3 Brownian Motion in Economy

Brownian movement's implications extend to economics, which analyzes people's economic decisions and their influence factors. The randomness of Brownian motion can be likened to the unpredictable nature of human behavior. Investors often exhibit irrational behaviors influenced by emotions and cognitive biases, leading to market anomalies that deviate from traditional economic theories.

Brownian movement plays a significant role in price fluctuation. Prices are dynamic. In contemporary society, the stochastic process is an important component of the market. Economists always describe price change by using the stochastic analytical framework. Economists make the security price at time t as $S(t)$. The gain or loss for the security in the economy is called "simple return." Simple return can be calculated by $S(t)-S(0)$, and the gain for the security without any influencing factor is called "natural return", and its formula is $(S(t)-S(0))/S(0)$. Economists always make the continuous rate of return as $X(t)$, and one has the fomula $X(t)=lnS(t)-lnS(0)$. The financial modeling document was defined the quantity X (t) as a stochastic process that represents the dynamic return [8].

Continuous movement and normal returns are fundamental and significant in economic study, which form the basic core of three theories, including option pricing theory, fundamental asset pricing methods, and portfolio theory. Portfolio theory emerged in the 1960s, which was the time between the demonstration of the random walk and the proposal of the efficient market hypothesis in practicing managing assets. This theoretical improvement has played an important role in the investment portfolio management industry: It motivated portfolio managers to try their best to pursue maximum diversification of assets and to widely benchmark against normal prices. The "benchmark" refers to a market index composed of securities representing certain aspects of the overall market. This widespread indexing is normal in contemporary society, but it also has some drawbacks, including limitations in asset management.

Due to Quetelet's shadow of the mean, it is related to the Brownian representation of dynamic return. This influence could be called the "Quetelet effect on managing asset methods." Markowitz and Sharpe accepted these properties and verified the Bachelier-Osborne representation for the concept of the optimal mean-variance portfolio [8].

Obviously, no one can be sure about finances in the future. Therefore, how to assume finance and make the best choice of investment becomes an important question for managers. To solve such a problem, it is significant to predict a stochastic process for all series of securities returns at each specific time. That is because calculating the variance-covariance matrix requires the characters of the movements of these securities in the form of probability vectors in the entire market. That means people need to have a joint distribution of the returns of all securities. The assumption of non-normality of security price changes makes the calculation to be possible, which provides the possibility for constructing the model of Markowitz's mean-variance optimal portfolio. Thus, the foundations of quantitative investment management methods come from academic studies on the Brownian representation of price changes and the randomness of stock market prices.

4. Conclusion

This paper discusses the basic properties and applications of random walk and Brownian motion. The author talks about the history of random walk and Brownian motion; then, the author shows the properties and functions of Brownian motion. After that, the author listed the applications for Brownian motion in three parts, including biology, chemistry, and economy. This paper shows many models that apply Brownian motion. Dynamic Brownian Bridge Movement Models are used to detect where animals are and where they may be gone. Brownian Thermal Ratchet is caused by the Brownian motion of each particles causing the phenomenon of non-equilibrium fluctuations leading to induce mechanical force and motion. Optical tweezer is one of the most significant applications for Brownian motion in chemistry, which makes it easier to understand the reaction. Brownian motion is also used for price fluctuation, whose core is the random walk, the same as Brownian motion. Generally, Brownian motion is popular in science and people's daily life.

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