Dean&Francis

ISSN 2959-6157

Heat Kernel estimations for the simple random walk

Xiaoyuan Wang

Nanjing Foreign Language School, Nanjing 210008, China

Abstract:

In this paper, we investigate the convergence rate of the heat kernel defined on locally finite graphs in the one-dimensional simple random walk case. The heat kernel represents the transition density and can be estimated by approximating the factorial. Through asymptotic expansions of certain functions and Taylor expansions of some terms, upper and lower bounds of the heat kernel can be derived.

Keywords: Random Walk; Heat Kernel; Markov Kernel; Markov Chain;

1. Notions and Background

Let V be a set of elements and $\mathbb{R}_+ := [0,\infty)$ be the set of all non-negative real numbers. A *Markov Kernel* on V is defined as

$$P(x,y): V\times V \to \mathbb{R}_+,$$

where

$$\sum_{y\in V} P(x,y) = 1, ?\forall x \in V.$$

Consider a sequence of random variables $\{X_n\}_{n=0}^{\infty}$. For a Markov kernel p, if X_n satisfies that

$$\mathbb{P}(X_{n+1} = y | X_n = x) = P(x,y), ? n \in \mathbb{N},$$

the *Markov property* (the process behavior at the moment n onward is independent of the past) is satisfied. Hence, $\{X_n\}_{n=0}^{\infty}$ is a *Markov chain* induced by p. It would be amusing to explore Markov chains on a finite graph (V,E), where V consists of finite number of vertices and E is the set of edges. A *simple random walk* X_n is defined as follows: for a walker at position $X_n \in V$ at time $n\in\mathbb{N}$, X_{n+1} is obtained by

$$\mathbb{P}(X_{n+1} = y | X_n = x) = \begin{cases} \frac{1}{deg(x)}, y \sim x, \\ 0, y \sim ? x, \end{cases}$$

where $deg(x)$ denotes the degree of $x \in V$, i.e., the number of edges that are attached to the vertex. To fully illustrate the probability pattern of simple random walk, we would like to introduce a probability measure on a set V^n. For points $x_1, x_2, \cdots, x_n \in V^n$, we define

(1.1)

$$\mathbb{P}_x^{(n)}(x_1, x_2, \cdots, x_n) := P(x, x_1) P(x_1, x_2) \cdots P(x_{n-1}, x_n).$$

If $(x_1, x_2, \cdots, x_n)$ is regarded as a n-points subset of V^n [1], then we can extend (1.1) to all $A \subset V^n$ by

$$(1.2) \quad \mathbb{P}_x^{(n)}(A) = \sum_{(x_1, x_2, \cdots, x_n)\in A} \mathbb{P}_x^{(n)}(x_1, x_2, \cdots, x_n).$$

It can be verified by induction that (see [1])

$$\mathbb{P}_x^{(n)}(V^n) = 1.$$

To extend (1.2) to V^∞, we need to use the Ionescu-Tulcea theorem.

Theorem 1.1 ([2]). *Let* $(\Omega_0, \mathcal{A}_0, P_0)$ *be a probability*

ISSN 2959-6157

space and $(\Omega_i,\mathcal{A}_i)$, $i\geqslant 1$, *be measurable spaces, where* Ω_i, $i\in\mathbb{N}$, *are sample spaces,* $\mathcal{A}_i$ *is a* σ*-field of* Ω_i, *and* P_0 *is a probablity measure. For each* $i\in\mathbb{N}$, *let*

$$\kappa_i:(\Omega^{i-1},\mathcal{A}^{i-1})\to(\Omega_i,\mathcal{A}_i)$$

be the Markov kernel derived from $(\Omega^{i-1},\mathcal{A}^{i-1})$ *and* $(\Omega_i,\mathcal{A}_i)$, *where*

$$\Omega^i:=\prod_{k=0}^{i}\Omega_k$$

and

$$\mathcal{A}^i:=?_{k=0}^{i}\mathcal{A}_k.$$

Then there exists a sequence of probability measures

$$P_i:=P_0\otimes ?_{k=1}^{i}\kappa_k$$

defined on the product space for the sequence $(\Omega^i,\mathcal{A}^i)$, $i\in\mathbb{N}$, *and there exists a uniquely defined probability measure P on* $\left(\prod_{k=0}^{\infty}\Omega_k, ?_{k=0}^{\infty}\mathcal{A}_k\right)$, *so that*

$$P_i(A)=P\left(A\times\prod_{k=i+1}^{\infty}\Omega_k\right)$$

is satisfied for each $A\in\mathcal{A}^i$ *and* $i\in\mathbb{N}$, *which means that the measure P has conditional probabilities equal to the stochastic kernels.*

Proposition 1.2 ([1]).

$$\mathbb{P}_x^{(n)}(x_1,\cdots,x_n)=\mathbb{P}_x^{(m)}(x_1,\cdots,x_n),$$

where $m>n$ and $m,n\in\mathbb{N}^*$.

Proof. For $m>n$, let $\mathbf{x}=(x_1,\cdots,x_n)$, $\mathbf{y}=(x_1,\cdots,x_m)$, and denote $\mathbf{y}/\mathbf{x}=(x_{n+1},\cdots,x_m)$, then

$$\begin{aligned}\mathbb{P}_x^{(m)}(\mathbf{y}) &= \sum_{\mathbf{y}/\mathbf{x}\in V}\mathbb{P}_x^{(m)}(\mathbf{y})=\sum_{\mathbf{y}/\mathbf{x}\in V}P(x,x_1)\prod_{i=1}^{m}P(x_i,x_i+1)\\ &= P(x,x_1)\prod_{i=1}^{n}P(x_i,x_i+1)\sum_{\mathbf{y}/\mathbf{x}\in V}\prod_{i=n+1}^{m}P(x_i,x_{i+1})\\ &= \mathbb{P}_x^{(n)}(x_1,\cdots,x_n)\mathbb{P}_{x_n}^{(n-m)}(V^{n-m})\\ &= \mathbb{P}_x^{(n)}(x_1,\cdots,x_n).\end{aligned}$$

It follows from Proposition (1.2) that

$$\mathbb{P}_x^{(n)}(A)=\mathbb{P}_x^{(m)}(A'),$$

here $(x_1,\cdots,x_m)\in A'$ if $(x_1,\cdots,x_n)\in A$.

Now we set

$$\mathbb{P}_x(A')=\mathbb{P}_x^{(n)}(A)$$

on V^∞, where

$$A'=\left\{\{x_k\}_{k=1}^{\infty}:(x_1,\cdots,x_n)\in A\right\}.$$

By the Ionescu-Tulcea theorem, this $\mathbb{P}_x$ defined on V^∞ is a unique probability measure.

Define X_n on the probability space Ω by

$$X_n\left(\{x_k\}_{k=1}^{\infty}\right)=x_n,$$

then (1.1) can be rewritten as

(1.3)

$$\mathbb{P}_x(X_1=x_1,\cdots,X_n=x_n)=P(x,x_1)P(x_1,x_2)\cdots P(x_{n-1},x_n).$$

Given a Markov chain $\{X_n\}$ with a Markov kernel $P(x,y)$, by (1.3)

$$P(x,y)=\mathbb{P}_x(X_1=y),$$

then $P(x,\cdot)$ is the distribution of $X_1, P_n(x,\cdot)$ is the distribution of X_n, here

$$P_n(x,y)=\mathbb{P}_x(X_n=y),$$

which also describes the random walk at time n, thereby $P_n(x,y)$ is also called the *n-step transition function*. Here, the reversibility of the transition function is defined as follows.

Definition 1.3. Let $P(x,y)$ be a Markov kernel. If there exists a positive function $\mu(x)$ on V such that

(1.4) $\quad P(x,y)\mu(x)=P(y,x)\mu(y),$

then $P(x,y)$ is called be *reversible*.

By the definition of $P_n(x,y)$, we have

(1.5) $\quad P_{n+1}(x,y)=\sum_{z\in V}P_n(x,z)P(z,y),$

then one can easily obtain from (1.4) and (1.5) by induction that Markov kernel $P_n(x,y)$ is reversible. Then, based on the reversibility, the heat kernel is defined.

Definition 1.4. Let V be a set of vertices with weights μ, then for the locally finite (V,μ),

$$p_n(x,y):=\frac{P_n(x,y)}{\mu(y)}$$

is called the *heat kernel* of (V,μ).

In the sense of random walk, the heat kernel is also called the *transition density* of the random walk. By definition (1.4), $\forall x,y\in\mathbb{Z}$,

$$p_n(x,y)=p_n(y,x).$$

Let (V,μ) be $\mathbb{Z}$ with a simple weight, then

$$p_n(x,y)=\frac{1}{2}P_n(x,y),$$

since $\forall x\in\mathbb{Z}$, $\mu(x)=2$, which is the case of one-dimensional simple random walk.

Since $\mathbb{Z}$ is shift invariant, $p_n(x,y)$ is also shift invariant, then

$$p_n(x,y)=p_n(x-z,y-z)$$

for $\forall z\in\mathbb{Z}$.

In particular, for $z=x$,

$$p_n(x,y)=p_n(0,y-x).$$

Therefore we only investigate $p_n(0,x)$ in the following.

Theorem 1.5. *Under the definition given by* (1.4) *and the symmetry* $p_n(x,y)=p_n(y,x)$ *for all* $x,y\in\mathbb{Z}$, *we have*

$$p_n(0,x)=\begin{cases}\frac{1}{2^{n+1}}\begin{pmatrix}n\\ \frac{n+x}{2}\end{pmatrix}, x\equiv n(mod 2), and |x|\leqslant n,\\ 0,?\text{otherwise},\end{cases}$$

where $\begin{pmatrix}n\\ m\end{pmatrix}$ *is the binomial coefficient.*

Proof. Since $p_n(0,x) = \frac{1}{2}P_n(0,x)$ for the one-dimensional simple random walk on $\mathbb{Z}$, we only need to prove that

$$\mathbb{P}_0(X_n=x)=P_n(0,x)=\frac{1}{2^n}\begin{pmatrix}n\\ \frac{x+n}{2}\end{pmatrix}.$$

In fact,

$$\begin{aligned}\mathbb{P}_0(X_n=x) &= \sum_{x_1,x_2,\dots,x_n\in V}\mathbb{P}_0(X_1=x_1,X_2=x_2,\cdots,X_n=x)\\ &= \sum_{x_1,x_2,\dots,x_n\in\mathbb{Z}}P(0,x_1)P(x_1,x_2)\cdots P(x_{n-1},x).\end{aligned}$$

Let

$$x_1-0=d_1, x_2-x_1=d_2,\cdots,x_{n-1}-x_{n-2}=d_{n-1}, x-x_{n-1}=d_n,$$

then $\mathbb{P}_0(X_n=x)\cdot 2^n$ is equal to the number of solutions to the equation

(1.6) $\quad d_1+d_2+\cdots+d_n=x,$

where for $\forall 1\leqslant i\leqslant n(i\in\mathbb{N}), d_i\in\{-1,1\}$.

Note that for $\forall 1\leqslant j\leqslant k\leqslant n(j,k\in\mathbb{N})$,

$$d_i\equiv d_j\equiv 1(mod 2),$$

$$|d_i|=|d_j|=1,$$

then

$$\sum_{i=1}^n d_i\equiv n(mod 2),$$

$$\left|\sum_{i=1}^n d_i\right|\leqslant\sum_{i=1}^n|d_i|=n,$$

whence the solutions to the equations exist if and only if $|x|\leqslant n and x\equiv n(mod 2)$.

Let

$$n_+=\#\{i\in\mathbb{N}|d_i=1,1\leqslant i\leqslant n\},$$

$$n_-=\#\{j\in\mathbb{N}|d_j=-1,1\leqslant j\leqslant n\},$$

then

$$(n_+)+(n_-)=n,$$

$$(n_+)-(n_-)=x,$$

whence

$$n_+=\frac{n+x}{2},$$

$$n_-=\frac{n-x}{2}.$$

In conclusion, if $x\equiv n(mod 2)$ and $|x|\leqslant n$ holds, we have

$$\mathbb{P}_0(X_n=x)=\frac{1}{2^n}\begin{pmatrix}n\\ \frac{n+x}{2}\end{pmatrix}.$$

In this case, the total number of solutions to the equation (1.6) is $\begin{pmatrix}n\\ \frac{n+x}{2}\end{pmatrix}$ once the condition of x is satisfied. Otherwise, the solution does not exists.

As n grows larger, the value of the factorials is hard to derive, therefore estimates are needed. In the lecture notes [1], A. Grigor'yan proved an estimate of the heat kernel:

Theorem 1.6. *For all positive integers* n *and for all* $x\in\mathbb{Z}$ *such that* $|x|\leqslant n$ *and* $x\equiv n mod 2$, *the following inequalities hold:*

$$\frac{C_2}{\sqrt{n}}e^{-(log 2)\frac{x^2}{n}}\leqslant p_n(0,x)\leqslant\frac{C_1}{\sqrt{n}}e^{-\frac{x^2}{2n}}$$

where C_1, C_2 *are some positive constants.*

Due to the method used for the estimate in [1], C_1, C_2 cannot be clearly derived. To illustrate the value of the heat kernel more clearly for a large n, we introduce the following estimates, which are to some extent modifications of the result (1.7).

2. Main Results of Estimations

Here we give three estimates of the heat kernel, through which the rates of convergence towards their asymptotic values (the first term of each equation) can be observed. Within the available range, we obtained these three results, which can be applied according to different accuracy requirements.

Theorem 2.1. *Assuming that* $|x|\leqslant n and x\equiv n(mod 2)$, *we*

ISSN 2959-6157

have

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot P_1\cdot I_1\leqslant p_n(0,x)\leqslant\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot Q_1\cdot J_1,$$

where

$$P_1=1-\frac{x^2}{2n},$$

$$Q_1=1,$$

$$I_1=1-\frac{1}{4n}+\frac{60\pi^2-1}{360n^3\left(4\pi^2+1\right)}-\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\frac{8\pi^2-5n^2\left(4\pi^2+1\right)}{15n^5\left(4\pi^2+1\right)}\cdot x^2,$$

$$J_1=1-\frac{1}{4n}+\frac{15\pi^2+4}{90n^3\left(4\pi^2+1\right)}+\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\left(\frac{4}{15n^5}-\frac{1}{3n^3}\right)\cdot x^2,$$

and particularly,

$$\frac{1}{\sqrt{2\pi n}}\cdot\left(1-\frac{1}{4n}-\frac{15\pi^2+4}{90n^3\left(4\pi^2+1\right)}\right)\leqslant p_n(0,0)\leqslant\frac{1}{\sqrt{2\pi n}}\cdot\left(1-\frac{1}{4n}+\frac{1-60\pi^2}{360n^3\left(4\pi^2+1\right)}\right).$$

Theorem 2.2. *For* $n,x\in\mathbb{N},\ |x|\leqslant n and x\equiv n(mod 2)$,

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot P_2\cdot I_2\leqslant p_n(0,x)\leqslant\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot Q_2\cdot J_2,$$

where

$$P_2=1-\frac{x^2}{2n},$$

$$Q_2=1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3},$$

$$\begin{aligned}I_2 = {}& 1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7\pi^2+2}{315n^5\left(4\pi^2+1\right)}-\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x\\ &-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{2}{21n^7}\right)\cdot x^2-\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3\\ &-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{4}{9n^9}\right)\cdot x^4,\end{aligned}$$

$$\begin{aligned}J_2 = {}& 1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7}{1260n^5}+\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x\\ &-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{8\pi^2}{21n^7\left(4\pi^2+1\right)}\right)\cdot x^2+\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3\\ &-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{8\pi^2}{9n^9\left(4\pi^2+1\right)}\right)\cdot x^4,\end{aligned}$$

and particularly,

$$\frac{1}{\sqrt{2\pi n}}\cdot\left(1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{63\pi^2+16}{315n^5\left(4\pi^2+1\right)}\right)\leqslant p_n(0,0)\leqslant\frac{1}{\sqrt{2\pi n}}\cdot\left(1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{252\pi^2-1}{1260n^5}\right).$$

Theorem 2.3. *For* $n,x\in\mathbb{N},\ |x|\leqslant n and x\equiv n(mod 2)$,

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot P_3\cdot I_3\leqslant p_n(0,x)\leqslant\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot Q_3\cdot J_3,$$

where

$$P_3=1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}-\frac{\left(5n^2-10n+8\right)x^6}{240n^5},$$

$$Q_3=1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3},$$

$$\begin{aligned}I_3 = {}& 1-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}+\frac{31\pi^2+8}{420n^7\left(4\pi^2+1\right)}-\frac{1}{15n^8\left(4\pi^2+1\right)}\\ &-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{8}{15n^9}\right)\cdot x^2-\frac{4}{5n^{10}}\cdot x^3\\ &-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{4}{n^{11}}\right)\cdot x^4-\frac{22}{5n^{12}}\cdot x^5\\ &-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{88}{5n^{13}}\right)\cdot x^6,\end{aligned}$$

$$\begin{aligned}J_3 = {}& 1-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}-\frac{124\pi^2-1}{1680n^7\left(4\pi^2+1\right)}+\frac{1}{15n^8\left(4\pi^2+1\right)}\\ &-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{32\pi^2}{15n^9\left(4\pi^2+1\right)}\right)\cdot x^2+\left(\frac{4}{5n^{10}\left(4\pi^2+1\right)}\right)\cdot x^3\\ &-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{16\pi^2}{n^{11}\left(4\pi^2+1\right)}\right)\cdot x^4+\left(\frac{22}{5n^{12}\left(4\pi^2+1\right)}\right)\cdot x^5\\ &-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{352\pi^2}{5n^{13}\left(4\pi^2+1\right)}\right)\cdot x^6,\end{aligned}$$

and particularly,

$$\frac{1}{\sqrt{2\pi n}}\cdot\left(1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{1020\pi^2-1}{1680n^7\left(4\pi^2+1\right)}\right)\leqslant p_n(0,0)\leqslant\frac{1}{\sqrt{2\pi n}}\cdot\left(1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{255\pi^2+64}{420n^7\left(4\pi^2+1\right)}\right).$$

Corollary 2.4. Clearly, for a fixed x,

$$p_n(0,x) \to \frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}$$

when $n \to \infty$ and particularly,

$$p_n(0,0) \to \frac{1}{\sqrt{2\pi n}}$$

when $n \to \infty$.

3. Detailed Proof of Theorems

Lemma 3.1. *Denote* $\Gamma(\cdot)$ *as the well-known Gamma function*

$$\Gamma(z) := \int_{t\geqslant 0} t^{z-1}e^{-t}dt,$$

and let $\mu(\cdot)$ *be a function*

$$\mu(z) := \int_{t\geqslant 0} \frac{e^{-tz}}{t} f(t)dt,$$

where $f(t) := \left(e^t - 1\right)^{-1} - t^{-1} + 1/2$. *For any* $n \in \mathbb{N}^*$,

$$n! = \Gamma(n+1) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\mu(n)},$$

Proof. From the result of Dirichlet and Gauss, when $Re(z) > 0$, we have

$$\begin{aligned}\Psi(z) &= \int_{t\geqslant 0}\left(t^{-1}e^{-t} - (1+t)^{-3}\right)dt \\ &= \int_{x\geqslant 0}\left(x^{-1}e^{-x} - \left(1-e^{-x}\right)^{-1}e^{-zx}\right)dx,\end{aligned}$$

here the Ψ function is defined as

$$\Psi(z) := -\gamma - \sum_{n\geqslant 0}\left(\frac{1}{z+n} - \frac{1}{n+1}\right) = \sum_{n\geqslant 0}\left[log\left(1+\frac{1}{n}\right) - \frac{1}{z+n}\right],$$

where

$$\zeta(1) := \gamma,$$

here

$$\zeta(z) := \sum_{n\geqslant 1}\frac{1}{n^z}$$

is the Riemann ζ function.

Then we have

$$\begin{aligned}\Psi(z+1) &= \int_{x\geqslant 0}\left(\frac{e^{-x}}{x} - \frac{e^{-zx}}{e^x - 1}\right)dx \\ &= \int_{x\geqslant 0}\frac{1}{2}e^{-zx}dx + \int_{x\geqslant 0}\frac{e^{-x} - e^{-zx}}{2}dx \\ &\quad - \int_{x\geqslant 0} f(x)e^{-zx}dx \\ &= \frac{1}{2z} + logz - \int_{x\geqslant 0} f(x)e^{-zx}dx,\end{aligned}$$

whence

$$\Psi(z) = -\frac{1}{2z} + logz - \int_{x\geqslant 0} f(x)e^{-zx}dx,$$

where $\Psi(z+1) - \Psi(z) = \frac{1}{z}$ has been used.

Note that

$$\mu(z) == \int_{t\geqslant 0}\frac{e^{-tz}}{t} f(t)dt.$$

Then we have

$$\begin{aligned}log\Gamma(z+1) &= log\Gamma(z+1) - log\Gamma(2) \\ &= \int_1^z \Psi(\omega+1)d\omega \\ &= \frac{logz}{2} + z(logz - 1) + 1 + \mu(z) - \mu(1),\end{aligned}$$

whence

$$\Gamma(z+1) = z^{z+\frac{1}{2}}e^{1-z}e^{\mu(z)-\mu(1)} = \sqrt{2\pi n}\left(\frac{z}{e}\right)^z e^{\mu(z)}$$

,

where

$$\mu(1) = 1 - log\sqrt{2\pi}$$

has been used. Then

$$n! = \Gamma(n+1) = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\mu(n)}, \forall n \in \mathbb{N}^*.$$

Lemma 3.2.

$$\mu(z) = \sum_{1\leqslant k\leqslant m}\frac{(-1)^{k-1}B_{2k}}{(2k)(2k-1)\ z^{2k-1}} + (-1)^m \xi_m(z)\frac{B_{2m+2}}{(2m+2)(2m+1)z^{2m+1}},$$

where

$$B_n := \frac{2n!\zeta(n)}{(2\pi)^n}$$

is the n*-th Bernoulli number and* $\xi_m(z) \in (0,1)$ *when* $z \in \mathbb{R}$.

Proof. We borrow the definition of f in Lemma 11 in this proof. From

$$xcoth(x) = 1 + 2x^2\sum_{k\geqslant 1}\frac{1}{x^2 + (\pi n)^2}$$

and

$$f(x) = -\frac{1}{x} + \frac{1}{2}coth\frac{x}{2}\left(0 < |x| < 1\right),$$

we get

$$f(x)\frac{1}{x} = 2\sum_{k\geqslant 1}\frac{1}{x^2 + (2\pi n)^2}.$$

Also

ISSN 2959-6157

$$\frac{1}{x^2+(2\pi n)^2} = \frac{1}{(2\pi n)^2}\cdot\frac{1}{1+\frac{x^2}{(2\pi n)^2}}$$

$$= \frac{1}{(2\pi n)^2}\left[\sum_{0\leqslant k\leqslant m-1}(-1)^k\frac{x^{2k}}{(2\pi n)^{2k}}+(-1)^m\frac{r_n}{(2\pi n)^{2m}}x^{2m}\right],$$

where

$$r_n(x):=\frac{4n^2\pi^2}{4n^2\pi^2+x^2}\in(0,1].$$

Then for every $m\in\mathbb{N}$, we have

$$f(x)\frac{1}{x} = 2\sum_{1\leqslant k\leqslant m}\sum_{k\geqslant1}\frac{(-1)^{k-1}x^{2k-2}}{(2\pi n)^{2k}}$$

$$+2(-1)^m\theta_m(x)x^{2m}\sum_{k\geqslant1}\frac{1}{(2\pi n)^{2m+2}},$$

where

$$\theta_m(x)=\frac{\sum_{k\geq1}\frac{r_n}{(2\pi n)^{2m+2}}}{\sum_{k\geq1}\frac{1}{(2\pi n)^{2m+2}}}\in(0,1].$$

From

$$\zeta(2n)=\frac{(2\pi)^n B_{2n}}{2(2n)!},n\geqslant1,$$

where B_n denotes the Bernoulli number, $n\in\mathbb{N}$, and

$$\mu(z)=\int_{x\geqslant0}f(x)e^{-zx}\frac{dx}{x},$$

we get

$$\mu(z) = 2\sum_{1\leqslant k\leqslant m}\sum_{k\geqslant1}\frac{(-1)^{k-1}}{(2\pi n)^{2k}}\int_{x\geqslant0}x^{2k-2}e^{-zx}dx$$

$$+2(-1)^m\theta_m\sum_{n\in\mathbb{N}}\frac{1}{(2\pi n)^{2m+2}}\int_{x\geqslant0}x^{2m}e^{-zx}dx$$

$$= 2\sum_{1\leqslant k\leqslant m}\sum_{k\geqslant1}\frac{(-1)^{k-1}}{(2\pi n)^{2k}}\frac{(2k-2)!}{z^{2k-1}}+2(-1)^m\xi_m(z)\sum_{k\geqslant1}\frac{1}{(2\pi n)^{2m+2}}\frac{(2m)!}{z^{2m+1}}$$

$$= 2\sum_{1\leqslant k\leqslant m}\frac{(-1)^{k-1}(2k-2)!}{(2\pi)^{2k}z^{2k-1}}\zeta(2k)$$

$$+2(-1)^m\xi_m(z)\frac{(2m)!}{z^{2m+1}}\frac{1}{(2\pi)^{2m+2}}\zeta(2m+2)$$

$$= \sum_{1\leqslant k\leqslant m}\frac{(-1)^{k-1}B_{2k}}{(2k)(2k-1)z^{2k-1}}+(-1)^m\xi_m(z)\frac{B_{2m+2}}{(2m+2)(2m+1)z^{2m+1}},$$

where

$$\xi_m(z)=\frac{\int_0^\infty\theta_m(x)x^{2m}e^{-zx}dx}{\int_0^\infty x^{2m}e^{-zx}dx}.$$

Only the case of $z\in\mathbb{R}$ is associated with our case. When $z\in\mathbb{R}$, we have

$$\xi_m(z)\in(0,1],$$

and particularly,

$$\xi_m(z)=\frac{\int_0^\infty\theta_m(x)x^{2m}e^{-zx}dx}{\int_0^\infty x^{2m}e^{-zx}dx}\in\left[\frac{4\pi^2}{4\pi^2+1},1\right]$$

for $|x|\leqslant n$.

Now we begin the proof of the main theorems.

Take $m=1$ in Lemma 3.2, we obtain Theorem 2.1 as follows.

Proof. By Theorem 5 and Lemma 3.1

$$p_n(0,0) = \frac{1}{2^{n+1}}\frac{n!}{\left(\left(\frac{n}{2}\right)!\right)^2}$$

$$= \frac{1}{2^{n+1}}\frac{\sqrt{2\pi n}n^n e^{-n}e^{\mu(n)}}{\left(\sqrt{\pi n}\left(\frac{n}{2}\right)^{\frac{n}{2}}e^{-\frac{n}{2}}e^{\mu\left(\frac{n}{2}\right)}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi n}}\frac{1}{e^{2\mu\left(\frac{n}{2}\right)-\mu(n)}},$$

and by Lemma 3.2 (take $m=1$),

$$2\mu\left(\frac{n}{2}\right)-\mu(n)=\frac{1}{3n}-\xi_1\left(\frac{n}{2}\right)\frac{2}{45n^3}-\frac{1}{12n}+\xi_1(n)\frac{1}{360n^3},$$

where

$$\xi_1\left(\frac{n}{2}\right)=\frac{\int_0^\infty\frac{\sum_{k\geqslant1}\frac{4n^2\pi^2}{(4n^2\pi^2+x^2)(2\pi n)^4}}{\sum_{k\geqslant1}\frac{1}{(2\pi n)^4}}\left(x^2e^{-\frac{n}{2}x}\right)dx}{\int_0^\infty\left(x^2e^{-\frac{n}{2}x}\right)dx},$$

$$\xi_1(n)=\frac{\int_0^\infty\frac{\sum_{k\geqslant1}\frac{4n^2\pi^2}{(4n^2\pi^2+x^2)(2\pi n)^4}}{\sum_{k\geqslant1}\frac{1}{(2\pi n)^4}}\left(x^2e^{-nx}\right)dx}{\int_0^\infty\left(x^2e^{-\frac{n}{2}x}\right)dx},$$

here

$$\mu\left(\frac{n}{2}\right) = \sum_{1\leqslant k\leqslant m}\frac{(-1)^{k-1}B_{2k}}{2k(2k-1)\left(\frac{n}{2}\right)^{2k-1}}+(-1)^m\xi_m\frac{B_{2m+2}}{(2m+2)(2m+1)\left(\frac{n}{2}\right)^{2m+1}}$$

$$= \frac{B_2}{2\left(\frac{n}{2}\right)}-\xi_1\left(\frac{n}{2}\right)\frac{B_4}{12\left(\frac{n}{2}\right)^3}$$

$$= \frac{1}{6n}-\xi_1\left(\frac{n}{2}\right)\frac{1}{45n^3},$$

and

$$\begin{aligned}\mu(n) &= \sum_{1\leqslant k\leqslant m}\frac{(-1)^{k-1}B_{2k}}{2k(2k-1)n^{2k-1}}+(-1)^m\theta_m\frac{B_{2m+2}}{(2m+2)(2m+1)n^{2m+1}}\\ &= \frac{B_2}{2n}-\xi_1(n)\frac{B_4}{12n^3}\\ &= \frac{1}{12n}-\xi_1(n)\frac{1}{360n^3},\end{aligned}$$

we have

$$\begin{aligned}p_n(0,0) &= \frac{1}{\sqrt{2\pi n}}\frac{1}{e^{2\mu\left(\frac{n}{2}\right)-\mu(n)}}\\ &= \frac{exp\left(\frac{\xi_1(n)-16\xi_1\left(\frac{n}{2}\right)}{360n^3}-\frac{1}{4n}\right)}{\sqrt{2\pi n}}.\end{aligned}$$

Then

$$\begin{aligned}p_n(0,0) &\leqslant \frac{1}{\sqrt{2\pi n}}\cdot exp\left(\frac{1-\frac{64\pi^2}{4\pi^2+1}}{360n^3}-\frac{1}{4n}\right)\\ &= \frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}+\frac{1-60\pi^2}{360n^3(4\pi^2+1)}\right),\end{aligned}$$

$$\begin{aligned}p_n(0,0) &\geqslant \frac{1}{\sqrt{2\pi n}}\cdot exp\left(\frac{\frac{4\pi^2}{4\pi^2+1}-16}{360n^3}-\frac{1}{4n}\right)\\ &= \frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}-\frac{15\pi^2+4}{90n^3(4\pi^2+1)}\right),\end{aligned}$$

whence

$$\frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}-\frac{15\pi^2+4}{90n^3(4\pi^2+1)}\right)\leqslant p_n(0,0)\leqslant \frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}+\frac{1-60\pi^2}{360n^3(4\pi^2+1)}\right).$$

Using the Maclaurin series of the natural exponential function,

$$\begin{aligned}exp\left(-\frac{1}{4n}-\frac{15\pi^2+4}{90n^3(4\pi^2+1)}\right) &\approx 1-\frac{1}{4n}-\frac{15\pi^2+4}{90n^3(4\pi^2+1)},\\ exp\left(1-\frac{1}{4n}+\frac{1-60\pi^2}{360n^3(4\pi^2+1)}\right) &\approx 1-\frac{1}{4n}+\frac{1-60\pi^2}{360n^3(4\pi^2+1)},\end{aligned}$$

whence

$$\frac{1}{\sqrt{2\pi n}}\cdot\left(1-\frac{1}{4n}-\frac{15\pi^2+4}{90n^3(4\pi^2+1)}\right)\leqslant p_n(0,0)\leqslant\frac{1}{\sqrt{2\pi n}}\cdot\left(1-\frac{1}{4n}+\frac{1-60\pi^2}{360n^3(4\pi^2+1)}\right).$$

Similarly, for

$$\begin{aligned}p_n(0,x) &= \frac{1}{2^{n+1}}\binom{n}{\frac{x+n}{2}}\\ &= \frac{1}{2^{n+1}}\cdot\sqrt{2\pi n}n^ne^{-n}e^{\mu(n)}\\ &\quad\cdot\frac{1}{\sqrt{\pi(n+x)}\cdot\left(\frac{n+x}{2}\right)^{\frac{n+x}{2}}\cdot e^{\mu\left(\frac{n+x}{2}\right)}\cdot e^{-\frac{n+x}{2}}}\\ &\quad\cdot\frac{1}{\sqrt{\pi(n-x)}\cdot\left(\frac{n-x}{2}\right)^{\frac{n-x}{2}}\cdot e^{\mu\left(\frac{n-x}{2}\right)}\cdot e^{-\frac{n-x}{2}}}\\ &= \frac{n^n}{2^n\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\frac{1}{\left(\frac{n+x}{2}\right)^{\frac{n+x}{2}}\left(\frac{n-x}{2}\right)^{\frac{n-x}{2}}}\cdot e^{\mu(n)-\mu\left(\frac{n+x}{2}\right)-\mu\left(\frac{n-x}{2}\right)},\end{aligned}$$

notice that

$$\mu(n) = \frac{1}{12n}-\frac{\xi_1(n)}{360n^3},$$

$$\mu\left(\frac{n+x}{2}\right) = \frac{1}{6(n+x)}-\frac{\xi_1\left(\frac{n+x}{2}\right)}{45(n+x)^3},$$

$$\mu\left(\frac{n-x}{2}\right) = \frac{1}{6(n-x)}-\frac{\xi_1\left(\frac{n-x}{2}\right)}{45(n-x)^3},$$

and the Maclaurin series

$$\frac{1}{\left(\frac{n+x}{2}\right)^{\frac{n+x}{2}}\cdot\left(\frac{n-x}{2}\right)^{\frac{n-x}{2}}}=\frac{2^n}{n^n}\left(1-\frac{x^2}{2n}+\cdots\right),$$

$$\begin{aligned}\frac{1}{12n}-\frac{1}{6(n+x)}-\frac{1}{6(n-x)} &= -\frac{1}{4n}-\frac{x^2}{3n^3}-\cdots,\\ \frac{1}{45(n+x)^3} &= \frac{1}{45n^3}-\frac{x}{15n^4}+\frac{2x^2}{15n^5}+\cdots,\\ \frac{1}{45(n-x)^3} &= \frac{1}{45n^3}+\frac{x}{15n^4}+\frac{2x^2}{15n^5}+\cdots,\end{aligned}$$

we have

$$p_n(0,x)=\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\cdots\right)$$

ISSN 2959-6157

$$\cdot exp(-\frac{1}{4n}-\frac{x^2}{3n^3}+\cdots-\frac{\xi_1(n)}{360n^3}+\frac{\xi_1\left(\frac{n+x}{2}\right)+\xi_1\left(\frac{n-x}{2}\right)}{45n^3}$$

$$+\frac{\xi_1\left(\frac{n-x}{2}\right)-\xi_1\left(\frac{n+x}{2}\right)}{15n^4}\cdot x+\frac{\xi_1\left(\frac{n+x}{2}\right)+\xi_1\left(\frac{n-x}{2}\right)}{15n^5}\cdot 2x^2)$$

$$\approx\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}\right)$$

$$\cdot exp(-\frac{1}{4n}+\frac{8\left(\xi_1\left(\frac{n+x}{2}\right)+\xi_1\left(\frac{n-x}{2}\right)\right)-\xi_1(n)}{360n^3}+\frac{\xi_1\left(\frac{n+x}{2}\right)+\xi_1\left(\frac{n-x}{2}\right)}{45n^3}$$

$$+\frac{\xi_1\left(\frac{n-x}{2}\right)-\xi_1\left(\frac{n+x}{2}\right)}{15n^4}\cdot x+\left(\frac{2\left(\xi_1\left(\frac{n+x}{2}\right)+\xi_1\left(\frac{n-x}{2}\right)\right)}{15n^5}-\frac{1}{3n^3}\right)\cdot x^2).$$

Then

$$p_n(0,x)\leqslant$$

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot$$

$$exp\left(-\frac{1}{4n}+\frac{16-\frac{4\pi^2}{4\pi^2+1}}{360n^3}+\frac{1-\frac{4\pi^2}{4\pi^2+1}}{15n^4\left(4\pi^2+1\right)}\cdot x+\left(\frac{4}{15n^5}-\frac{1}{3n^3}\right)\cdot x^2\right)$$

$$=\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}$$

$$\cdot exp\left(-\frac{1}{4n}+\frac{15\pi^2+4}{90n^3\left(4\pi^2+1\right)}+\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\left(\frac{4}{15n^5}-\frac{1}{3n^3}\right)\cdot x^2\right),$$

$$p_n(0,x)\geqslant\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}\right)$$

$$\cdot exp\left(-\frac{1}{4n}+\frac{\frac{64\pi^2}{4\pi^2+1}-1}{360n^3}+\frac{\left(\frac{4\pi^2}{4\pi^2+1}-1\right)}{15n^4}\cdot x+\left(\frac{\frac{8\pi^2}{4\pi^2+1}}{15n^5}-\frac{1}{3n^3}\right)\cdot x^2\right)=$$

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}\right)$$

$$\cdot exp\left(-\frac{1}{4n}+\frac{60\pi^2-1}{360n^3\left(4\pi^2+1\right)}-\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\frac{8\pi^2-5n^2\left(4\pi^2+1\right)}{15n^5\left(4\pi^2+1\right)}\cdot x^2\right),$$

whence

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}\right)$$

$$\cdot exp\left(-\frac{1}{4n}+\frac{60\pi^2-1}{360n^3\left(4\pi^2+1\right)}-\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\frac{8\pi^2-5n^2\left(4\pi^2+1\right)}{15n^5\left(4\pi^2+1\right)}\cdot x^2\right)$$

$$\leqslant p_n(0,x)\leqslant$$

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}$$

$$\cdot exp\left(-\frac{1}{4n}+\frac{15\pi^2+4}{90n^3\left(4\pi^2+1\right)}+\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\left(\frac{4}{15n^5}-\frac{1}{3n^3}\right)\cdot x^2\right).$$

Using the Maclaurin series of the natural exponential function,

$$exp\left(-\frac{1}{4n}+\frac{60\pi^2-1}{360n^3\left(4\pi^2+1\right)}-\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\frac{8\pi^2-5n^2\left(4\pi^2+1\right)}{15n^5\left(4\pi^2+1\right)}\cdot x^2\right)$$

$$\approx 1-\frac{1}{4n}+\frac{60\pi^2-1}{360n^3\left(4\pi^2+1\right)}-\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\frac{8\pi^2-5n^2\left(4\pi^2+1\right)}{15n^5\left(4\pi^2+1\right)}\cdot x^2,$$

$$exp\left(-\frac{1}{4n}+\frac{15\pi^2+4}{90n^3\left(4\pi^2+1\right)}+\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\left(\frac{4}{15n^5}-\frac{1}{3n^3}\right)\cdot x^2\right)$$

$$\approx 1-\frac{1}{4n}+\frac{15\pi^2+4}{90n^3\left(4\pi^2+1\right)}+\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\left(\frac{4}{15n^5}-\frac{1}{3n^3}\right)\cdot x^2,$$

whence

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}\right)$$

$$\left(1-\frac{1}{4n}+\frac{60\pi^2-1}{360n^3\left(4\pi^2+1\right)}-\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\frac{8\pi^2-5n^2\left(4\pi^2+1\right)}{15n^5\left(4\pi^2+1\right)}\cdot x^2\right)$$

$$\leqslant p_n(0,x)\leqslant$$

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}$$

$$\cdot\left(1-\frac{1}{4n}+\frac{15\pi^2+4}{90n^3\left(4\pi^2+1\right)}+\frac{1}{15n^4\left(4\pi^2+1\right)}\cdot x+\left(\frac{4}{15n^5}-\frac{1}{3n^3}\right)\cdot x^2\right).$$

Similarly, take $m=2$ in Lemma 3.2, we obtain Theorem 2.2.

Proof. Here

$$2\mu\left(\frac{n}{2}\right)-\mu(n)$$

$$=\frac{1}{3n}-\frac{2}{45n^3}+\xi_2\left(\frac{n}{2}\right)\cdot\frac{16}{315n^5}-\frac{1}{12n}+\frac{1}{360n^3}-\xi_2(n)\cdot\frac{1}{1260n^5}$$

$$=\frac{1}{4n}-\frac{1}{24n^3}+\frac{64\xi_2\left(\frac{n}{2}\right)-\xi_2(n)}{1260n^5},$$

where

$$\xi_2\left(\frac{n}{2}\right)=\frac{\int_0^\infty \frac{\sum_{k\geqslant 1}\frac{r_k}{(2\pi k)^6}}{\sum_{k\geqslant 1}\frac{1}{(2\pi k)^6}} x^4 e^{-\frac{n}{2}x}dx}{\int_0^\infty x^4 e^{-\frac{n}{2}x}dx},\ \xi_2(n)=\frac{\int_0^\infty \frac{\sum_{k\geqslant 1}\frac{r_k}{(2\pi k)^6}}{\sum_{k\geqslant 1}\frac{1}{(2\pi k)^6}} x^4 e^{-nx}dx}{\int_0^\infty x^4 e^{-nx}dx},$$

here

$$\mu\left(\frac{n}{2}\right) = \sum_{1\leqslant k\leqslant 2}\frac{(-1)^{k-1}B_{2k}}{2k(2k-1)\left(\frac{n}{2}\right)^{2k-1}} +(-1)^2\xi_2\left(\frac{n}{2}\right)\frac{B_{2\cdot 2+2}}{(2\cdot 2+2)(2\cdot 2+1)\left(\frac{n}{2}\right)^{(2\cdot 2+1)}}$$

$$= \frac{1}{6n}-\frac{1}{45n^3}+\frac{8\xi_2\left(\frac{n}{2}\right)}{315n^5},$$

$$\mu(n) = \frac{1}{12n}-\frac{1}{360n^3}+\frac{\xi_2(n)}{1260n^5},$$

then we have

$$p_n(0,0)=\frac{1}{\sqrt{2\pi n}}\cdot e^{\mu(n)-2\mu\left(\frac{n}{2}\right)}$$

$$=\frac{1}{\sqrt{2\pi n}}\cdot exp\left(\frac{1-6n^2}{24n^3}-\frac{64\xi_2\left(\frac{n}{2}\right)-\xi_2(n)}{1260n^5}\right).$$

Then

$$p_n(0,0) \geqslant \frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{64-\frac{4\pi^2}{4\pi^2+1}}{1260n^5}\right)$$

$$= \frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{63\pi^2+16}{315n^5(4\pi^2+1)}\right),$$

$$p_n(0,0) \leqslant \frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{64\cdot\frac{4\pi^2}{4\pi^2+1}-1}{1260n^5}\right)$$

$$= \frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{252\pi^2-1}{1260n^5}\right),$$

whence

$$\frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{63\pi^2+16}{315n^5(4\pi^2+1)}\right) \leqslant p_n(0,0)\leqslant \frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{252\pi^2-1}{1260n^5}\right).$$

Using the Maclaurin series of the natural exponential function,

$$exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{63\pi^2+16}{315n^5(4\pi^2+1)}\right)\approx 1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{63\pi^2+16}{315n^5(4\pi^2+1)},$$

$$\frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{252\pi^2-1}{1260n^5}\right)\approx 1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{252\pi^2-1}{1260n^5},$$

whence

$$\frac{1}{\sqrt{2\pi n}}\cdot\left(1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{63\pi^2+16}{315n^5(4\pi^2+1)}\right) \leqslant p_n(0,0)\leqslant \frac{1}{\sqrt{2\pi n}}\cdot\left(1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{252\pi^2-1}{1260n^5}\right).$$

Similarly, for

$$p_n(0,x)=\frac{n^n}{2^n\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\frac{1}{\left(\frac{n+x}{2}\right)^{\frac{n+x}{2}}\left(\frac{n-x}{2}\right)^{\frac{n-x}{2}}}\cdot e^{\mu(n)-\mu\left(\frac{n+x}{2}\right)-\mu\left(\frac{n-x}{2}\right)},$$

notice that

$$\mu(n)=\frac{1}{12n}-\frac{1}{360n^3}+\frac{\xi_2(n)}{1260n^5},\ \mu\left(\frac{n+x}{2}\right)=\frac{1}{6(n+x)}-\frac{1}{45(n+x)^3}+\frac{8\xi_2\left(\frac{n+x}{2}\right)}{315(n+x)^5},$$

$$\mu\left(\frac{n-x}{2}\right)=\frac{1}{6(n-x)}-\frac{1}{45(n-x)^3}+\frac{8\xi_2\left(\frac{n-x}{2}\right)}{315(n-x)^5},$$

where

$$\xi_2(n)=\frac{\int_0^\infty \frac{\sum_{k\geqslant 1}\frac{r_k}{(2\pi k)^6}}{\sum_{k\geqslant 1}\frac{1}{(2\pi k)^6}} x^4 e^{-\frac{n-x}{2}\cdot x}dx}{\int_0^\infty x^4 e^{-\frac{n-x}{2}\cdot x}dx},$$

$$\xi_2\left(\frac{n+x}{2}\right)=\frac{\int_0^\infty \frac{\sum_{k\geqslant 1}\frac{r_k}{(2\pi k)^6}}{\sum_{k\geqslant 1}\frac{1}{(2\pi k)^6}} x^4 e^{-\frac{n+x}{2}\cdot x}dx}{\int_0^\infty x^4 e^{-\frac{n+x}{2}\cdot x}dx},$$

ISSN 2959-6157

$$\xi_2\left(\frac{n-x}{2}\right)=\frac{\int_0^{\infty}\frac{\sum_{k\geqslant 1}\frac{r_k}{(2\pi k)^6}}{\sum_{k\geqslant 1}\frac{1}{(2\pi k)^6}}x^4e^{-\frac{n-x}{2}\cdot x}dx}{\int_0^{\infty}x^4e^{-\frac{n-x}{2}\cdot x}dx},$$

and the Maclaurin series

$$\frac{1}{\left(\frac{n+x}{2}\right)^{\frac{n+x}{2}}\left(\frac{n-x}{2}\right)^{\frac{n-x}{2}}}=\frac{2^n}{n^n}\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}+\cdots\right),\frac{1}{12n}-$$

$$\frac{1}{6(n+x)}-\frac{1}{6(n-x)}=-\frac{1}{4n}-\frac{x^2}{3n^3}-\frac{x^4}{3n^5}+\cdots,$$

$$-\frac{1}{360n^3}+\frac{1}{45(n+x)^3}+\frac{1}{45(n-x)^3}=$$

$$\frac{1}{24n^3}+\frac{4x^2}{15n^5}+\frac{2x^4}{3n^7}+\cdots,$$

$$\frac{1}{315(n+x)^5}=\frac{1}{315n^5}-\frac{x}{63n^6}+\frac{x^2}{21n^7}-\frac{x^3}{9n^8}+\frac{2x^4}{9n^9}+\cdots,$$

$$\frac{1}{315(n-x)^5}=\frac{1}{315n^5}+\frac{x}{63n^6}+\frac{x^2}{21n^7}+\frac{x^3}{9n^8}+\frac{2x^4}{9n^9}+\cdots,$$

we have

$$p_n(0,x)\approx\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}\right)$$

$$\cdot exp(-\frac{1}{4n}-\frac{x^2}{3n^3}-\frac{x^4}{3n^5}+\frac{1}{24n^3}+\frac{4x^2}{15n^5}+\frac{2x^4}{3n^7}+\frac{\xi_2(n)}{1260n^5}-\frac{2}{315n^5}$$

$$+\frac{\xi_2\left(\frac{n+x}{2}\right)-\xi_2\left(\frac{n-x}{2}\right)}{63n^6}\cdot x-\frac{\xi_2\left(\frac{n+x}{2}\right)-\xi_2\left(\frac{n-x}{2}\right)}{21n^7}\cdot x^2$$

$$+\frac{\xi_2\left(\frac{n+x}{2}\right)-\xi_2\left(\frac{n-x}{2}\right)}{9n^8}\cdot x^3-\frac{2\left(\xi_2\left(\frac{n+x}{2}\right)-\xi_2\left(\frac{n-x}{2}\right)\right)}{9n^9}\cdot x^4).$$

$$=\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}\right)$$

$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{2}{315n^5}+\frac{\xi_2(n)}{1260n^5}$$

$$+\frac{\xi_2\left(\frac{n+x}{2}\right)-\xi_2\left(\frac{n-x}{2}\right)}{63n^6}\cdot x-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{\xi_2\left(\frac{n+x}{2}\right)+\xi_2\left(\frac{n-x}{2}\right)}{21n^7}\right)\cdot x^2$$

$$+\frac{\xi_2\left(\frac{n+x}{2}\right)-\xi_2\left(\frac{n-x}{2}\right)}{9n^8}\cdot x^3-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{2\left(\xi_2\left(\frac{n+x}{2}\right)+\xi_2\left(\frac{n-x}{2}\right)\right)}{9n^9}\right)\cdot x^4).$$

Then

$$p_n(0,x)\leqslant\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}\right)$$

$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7}{1260n^5}+\frac{1-\frac{4\pi^2}{4\pi^2+1}}{63n^6}\cdot x-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{\frac{8\pi^2}{4\pi^2+1}}{21n^7}\right)\cdot x^2$$

$$+\frac{1-\frac{4\pi^2}{4\pi^2+1}}{9n^8}\cdot x^3-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{\frac{8\pi^2}{4\pi^2+1}}{9n^9}\right)\cdot x^4)$$

$$=\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}\right)$$

$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7}{1260n^5}+\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x$$

$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{8\pi^2}{21n^7\left(4\pi^2+1\right)}\right)\cdot x^2+\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3$$

$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{8\pi^2}{9n^9\left(4\pi^2+1\right)}\right)\cdot x^4),$$

$$p_n(0,x)\geqslant\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}\right)$$

$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{2}{315n^5}+\frac{\frac{4\pi^2}{4\pi^2+1}}{1260n^5}+\frac{\frac{4\pi^2}{4\pi^2+1}-1}{63n^6}\cdot x$$

$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{2}{21n^7}\right)\cdot x^2+\frac{\frac{4\pi^2}{4\pi^2+1}-1}{9n^8}\cdot x^3-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{4}{9n^9}\right)\cdot x^4)$$

$$=\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}\right)$$

$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7\pi^2+2}{315n^5\left(4\pi^2+1\right)}-\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x$$

$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{2}{21n^7}\right)\cdot x^2-\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{4}{9n^9}\right)\cdot x^4),$$

whence

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}\right)$$

$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7\pi^2+2}{315n^5\left(4\pi^2+1\right)}-\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x$$

$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{2}{21n^7}\right)\cdot x^2-\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{4}{9n^9}\right)\cdot x^4),$$

$$\leqslant p_n(0,x)\leqslant$$

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}\right)$$
$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7}{1260n^5}+\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{8\pi^2}{21n^7\left(4\pi^2+1\right)}\right)\cdot x^2+\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{8\pi^2}{9n^9\left(4\pi^2+1\right)}\right)\cdot x^4).$$

Using the Maclaurin series of the natural exponential function,

$$exp(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7\pi^2+2}{315n^5\left(4\pi^2+1\right)}-\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{2}{21n^7}\right)\cdot x^2-\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{4}{9n^9}\right)\cdot x^4)$$
$$\approx \quad 1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7\pi^2+2}{315n^5\left(4\pi^2+1\right)}-\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{2}{21n^7}\right)\cdot x^2-\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{4}{9n^9}\right)\cdot x^4,$$

$$exp(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7}{1260n^5}+\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{8\pi^2}{21n^7\left(4\pi^2+1\right)}\right)\cdot x^2+\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{8\pi^2}{9n^9\left(4\pi^2+1\right)}\right)\cdot x^4)$$
$$\approx \quad 1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7}{1260n^5}+\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{8\pi^2}{21n^7\left(4\pi^2+1\right)}\right)\cdot x^2+\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{8\pi^2}{9n^9\left(4\pi^2+1\right)}\right)\cdot x^4,$$

whence

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}\right)$$
$$\cdot(1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7\pi^2+2}{315n^5\left(4\pi^2+1\right)}-\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{2}{21n^7}\right)\cdot x^2-\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{4}{9n^9}\right)\cdot x^4)$$
$$\leqslant p_n(0,x)\leqslant$$

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}\right)$$
$$\cdot(1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{7}{1260n^5}+\frac{1}{63n^6\left(4\pi^2+1\right)}\cdot x$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}+\frac{8\pi^2}{21n^7\left(4\pi^2+1\right)}\right)\cdot x^2+\frac{1}{9n^8\left(4\pi^2+1\right)}\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}+\frac{8\pi^2}{9n^9\left(4\pi^2+1\right)}\right)\cdot x^4).$$

Take $m=3$ in Lemma 3.2, we can obtain Theorem 2.3.

Proof. Here

$$2\mu\left(\frac{n}{2}\right)-\mu(n)$$
$$=\left(\frac{1}{3n}-\frac{2}{45n^3}+\frac{16}{315n^5}+\frac{16\xi_3\left(\frac{n}{2}\right)}{105n^7}\right)-\left(\frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}+\frac{\xi_3(n)}{1680n^7}\right)$$
$$=\frac{1}{4n}-\frac{1}{24n^3}+\frac{63}{1260n^5}+\frac{256\xi_3\left(\frac{n}{2}\right)-\xi_3(n)}{1680n^7},$$

where

$$\xi_3\left(\frac{n}{2}\right)=\frac{\int_0^\infty\frac{\sum_{k\geqslant1}\frac{r_k}{(2\pi k)^8}}{\sum_{k\geqslant1}\frac{1}{(2\pi k)^8}}x^6e^{-\frac{n}{2}x}dx}{\int_0^\infty x^6e^{-\frac{n}{2}\cdot x}dx}\quad \xi_3(n)=\frac{\int_0^\infty\frac{\sum_{k\geqslant1}\frac{r_k}{(2\pi k)^8}}{\sum_{k\geqslant1}\frac{1}{(2\pi k)^8}}x^6e^{-nx}dx}{\int_0^\infty x^6e^{-nx}dx},$$

here

$$\mu\left(\frac{n}{2}\right)=\sum_{1\leqslant k\leqslant3}\frac{(-1)^{k-1}B_{2k}}{2k(2k-1)\left(\frac{n}{2}\right)^{2k-1}}+(-1)^3\xi_3\left(\frac{n}{2}\right)\frac{B_{2\cdot3+2}}{(2\cdot3+2)(2\cdot3+1)}\left(\frac{n}{2}\right)^{(2\cdot3+1)}$$
$$=\frac{1}{6n}-\frac{1}{45n^3}+\frac{8}{315n^5}+\frac{8\xi_3\left(\frac{n}{2}\right)}{105n^7},$$

$$\mu(n)=\frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}+\frac{\xi_3(n)}{1680n^7},$$

we have

$$p_n(0,0)=\frac{1}{\sqrt{2\pi n}}\cdot e^{\mu(n)-2\mu\left(\frac{n}{2}\right)}$$
$$=\frac{1}{\sqrt{2\pi n}}\cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{256\xi_3\left(\frac{n}{2}\right)-\xi_3(n)}{1680n^7}\right).$$

Then

ISSN 2959-6157

$$p_n(0,0) \leqslant \frac{1}{\sqrt{2\pi n}} \cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{256-\frac{4\pi^2}{4\pi^2+1}}{1680n^7}\right)$$

$$= \frac{1}{\sqrt{2\pi n}} \cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{255\pi^2+64}{420n^7\left(4\pi^2+1\right)}\right),$$

$$p_n(0,0) \geqslant \frac{1}{\sqrt{2\pi n}} \cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{256\cdot\frac{4\pi^2}{4\pi^2+1}-1}{1680n^7}\right)$$

$$= \frac{1}{\sqrt{2\pi n}} \cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{1020\pi^2-1}{1680n^7\left(4\pi^2+1\right)}\right),$$

whence

$$\frac{1}{\sqrt{2\pi n}} \cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{1020\pi^2-1}{1680n^7\left(4\pi^2+1\right)}\right) \leqslant$$

$$p_n(0,0) \leqslant \frac{1}{\sqrt{2\pi n}} \cdot exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{255\pi^2+64}{420n^7\left(4\pi^2+1\right)}\right).$$

Using the Maclaurin series of the natural exponential function,

$$exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{1020\pi^2-1}{1680n^7\left(4\pi^2+1\right)}\right)$$

$$\approx 1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{1020\pi^2-1}{1680n^7\left(4\pi^2+1\right)},$$

$$exp\left(-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{255\pi^2+64}{420n^7\left(4\pi^2+1\right)}\right)$$

$$\approx 1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{255\pi^2+64}{420n^7\left(4\pi^2+1\right)},$$

whence

$$\frac{1}{\sqrt{2\pi n}} \cdot \left(1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{1020\pi^2-1}{1680n^7\left(4\pi^2+1\right)}\right)$$

$$\leqslant p_n(0,0) \leqslant$$

$$\frac{1}{\sqrt{2\pi n}} \cdot \left(1-\frac{1}{4n}+\frac{1}{24n^3}-\frac{1}{20n^5}+\frac{255\pi^2+64}{420n^7\left(4\pi^2+1\right)}\right).$$

Similarly, for

$$p_n(0,x) = \frac{n^n}{2^n\sqrt{\pi\left(n-\frac{x^2}{n}\right)}} \cdot \frac{1}{\left(\frac{n+x}{2}\right)^{\frac{n+x}{2}}\left(\frac{n-x}{2}\right)^{\frac{n-x}{2}}} \cdot$$

$$e^{\mu(n)-\mu\left(\frac{n+x}{2}\right)-\mu\left(\frac{n-x}{2}\right)},$$

notice that

$$\mu(n) = \frac{1}{12n}-\frac{1}{360n^3}+\frac{1}{1260n^5}+\frac{\xi_3(n)}{1680n^7},$$

$$\mu\left(\frac{n+x}{2}\right) = \frac{1}{6(n+x)}-\frac{1}{45(n+x)^3}+\frac{1}{315(n+x)^5}+\frac{8\xi_3\left(\frac{n+x}{2}\right)}{105(n+x)^7},$$

$$\mu\left(\frac{n-x}{2}\right) = \frac{1}{6(n-x)}-\frac{1}{45(n-x)^3}+\frac{1}{315(n-x)^5}+\frac{8\xi_3\left(\frac{n-x}{2}\right)}{105(n-x)^7},$$

where

$$\xi_3(n) = \frac{\int_0^\infty \frac{\sum_{k\geqslant1}\frac{r_k}{(2\pi k)^8}}{\sum_{k\geqslant1}\frac{1}{(2\pi k)^8}} x^6 e^{-nx}dx}{\int_0^\infty x^6 e^{-nx}dx},$$

$$\xi_3\left(\frac{n+x}{2}\right) = \frac{\int_0^\infty \frac{\sum_{k\geqslant1}\frac{r_k}{(2\pi k)^8}}{\sum_{k\geqslant1}\frac{1}{(2\pi k)^8}} x^6 e^{-\frac{n+x}{2}}dx}{\int_0^\infty x^6 e^{-\frac{n+x}{2}}dx},$$

$$\xi_3\left(\frac{n-x}{2}\right) = \frac{\int_0^\infty \frac{\sum_{k\geqslant1}\frac{r_k}{(2\pi k)^8}}{\sum_{k\geqslant1}\frac{1}{(2\pi k)^8}} x^6 e^{-\frac{n-x}{2}}dx}{\int_0^\infty x^6 e^{-\frac{n-x}{2}}dx},$$

and the Maclaurin series

$$\frac{1}{\left(\frac{n+x}{2}\right)^{\frac{n+x}{2}}\left(\frac{n-x}{2}\right)^{\frac{n-x}{2}}} =$$

$$\frac{2^n}{n^n}\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}-\frac{\left(5n^2-10n+8\right)x^6}{240n^5}+\cdots\right),$$

$$\frac{1}{12n}-\frac{1}{6(n+x)}-\frac{1}{6(n-x)} = -\frac{1}{4n}-\frac{x^2}{3n^3}-\frac{x^4}{3n^5}-\frac{x^6}{3n^7}+\cdots,$$

$$-\frac{1}{360n^3}+\frac{1}{45(n+x)^3}+\frac{1}{45(n-x)^3} = \frac{1}{24n^3}+\frac{4x^2}{15n^5}+\frac{2x^4}{3n^7}+\frac{56x^6}{45n^9}+\cdots,$$

$$-\frac{1}{1260n^5}-\frac{1}{315(n+x)^5}-\frac{1}{315(n-x)^5} = \frac{1}{180n^5}+\frac{2x^2}{21n^7}+\frac{4x^4}{9n^9}+\frac{4x^6}{3n^{11}}+\cdots,$$

$$\frac{1}{105(n+x)^7} = \frac{1}{105n^7}-\frac{x}{15n^8}+\frac{4x^2}{15n^9}-\frac{4x^3}{5n^{10}}+\frac{2x^4}{n^{11}}-\frac{22x^5}{5n^{12}}+\frac{44x^6}{5n^{13}}+\cdots,$$

we have

$$p(0,x) \approx \frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}} \cdot \left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}-\frac{\left(5n^2-10n+8\right)x^6}{240n^5}\right)$$

$$\cdot exp(-\frac{1}{4n}-\frac{x^2}{3n^3}-\frac{x^4}{3n^5}-\frac{x^6}{3n^7}+\frac{1}{24n^3}+\frac{4x^2}{15n^5}+\frac{2x^4}{3n^7}+\frac{56x^6}{45n^9}$$
$$+\frac{1}{180n^5}+\frac{2x^2}{21n^7}+\frac{4x^4}{9n^9}+\frac{4x^6}{3n^{11}}+\frac{\xi_3(n)}{1680n^7}-\frac{\xi_3\left(\frac{n+x}{2}\right)+\xi_3\left(\frac{n-x}{2}\right)}{105n^7}$$
$$+\frac{\xi_3\left(\frac{n-x}{2}\right)-\xi_3\left(\frac{n+x}{2}\right)}{15n^8}-\frac{4x^2\left(\xi_3\left(\frac{n+x}{2}\right)+\xi_3\left(\frac{n-x}{2}\right)\right)}{15n^9}$$
$$+\frac{4x^3\left(\xi_3\left(\frac{n-x}{2}\right)-\xi_3\left(\frac{n+x}{2}\right)\right)}{5n^{10}}-\frac{2x^4\left(\xi_3\left(\frac{n+x}{2}\right)+\xi_3\left(\frac{n-x}{2}\right)\right)}{n^{11}}$$
$$+\frac{22x^5\left(\xi_3\left(\frac{n-x}{2}\right)-\xi_3\left(\frac{n+x}{2}\right)\right)}{5n^{12}}-\frac{44x^6\left(\xi_3\left(\frac{n+x}{2}\right)+\xi_3\left(\frac{n-x}{2}\right)\right)}{5n^{13}})$$
$$=\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}-\frac{\left(5n^2-10n+8\right)x^6}{240n^5}\right)$$
$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}+\frac{\xi_3(n)}{1680n^7}-\frac{\xi_3\left(\frac{n+x}{2}\right)+\xi_3\left(\frac{n-x}{2}\right)}{105n^7}$$
$$+\frac{\xi_3\left(\frac{n-x}{2}\right)-\xi_3\left(\frac{n+x}{2}\right)}{15n^8}-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{4\left(\xi_3\left(\frac{n+x}{2}\right)-\xi_3\left(\frac{n+x}{2}\right)\right)}{15n^9}\right)\cdot x^2$$
$$+\left(\frac{4\left(\xi_3\left(\frac{n-x}{2}\right)-\xi_3\left(\frac{n+x}{2}\right)\right)}{5n^{10}}\right)\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{2\left(\xi_3\left(\frac{n+x}{2}\right)+\xi_3\left(\frac{n-x}{2}\right)\right)}{n^{11}}\right)\cdot x^4$$
$$+\left(\frac{22\left(\xi_3\left(\frac{n-x}{2}\right)-\xi_3\left(\frac{n+x}{2}\right)\right)}{5n^{12}}\right)\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{44\left(\xi_3\left(\frac{n+x}{2}\right)+\xi_3\left(\frac{n-x}{2}\right)\right)}{5n^{13}}\right)\cdot x^6).$$

Then

$$p_n(0,x)\le\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}\right)$$
$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}+\frac{1}{1680n^7}-\frac{\frac{8\pi^2}{4\pi^2+1}}{105n^7}+\frac{1-\frac{4\pi^2}{4\pi^2+1}}{15n^8}$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{\frac{32\pi^2}{4\pi^2+1}}{15n^9}\right)\cdot x^2+\left(\frac{4\left(1-\frac{4\pi^2}{4\pi^2+1}\right)}{5n^{10}}\right)\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{\frac{16\pi^2}{4\pi^2+1}}{n^{11}}\right)\cdot x^4+\left(\frac{22\left(1-\frac{4\pi^2}{4\pi^2+1}\right)}{5n^{12}}\right)\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{\frac{352\pi^2}{4\pi^2+1}}{5n^{13}}\right)\cdot x^6)$$
$$=\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}\right)$$
$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}-\frac{124\pi^2-1}{1680n^7\left(4\pi^2+1\right)}+\frac{1}{15n^8\left(4\pi^2+1\right)}$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{32\pi^2}{15n^9\left(4\pi^2+1\right)}\right)\cdot x^2+\left(\frac{4}{5n^{10}\left(4\pi^2+1\right)}\right)\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{16\pi^2}{n^{11}\left(4\pi^2+1\right)}\right)\cdot x^4+\left(\frac{22}{5n^{12}\left(4\pi^2+1\right)}\right)\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{352\pi^2}{5n^{13}\left(4\pi^2+1\right)}\right)\cdot x^6),$$

$$p_n(0,x)\geqslant\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}-\frac{\left(5n^2-10n+8\right)x^6}{240n^5}\right)$$
$$=\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}-\frac{\left(5n^2-10n+8\right)x^6}{240n^5}\right)$$
$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}+\frac{31\pi^2+8}{420n^7\left(4\pi^2+1\right)}-\frac{1}{15n^8\left(4\pi^2+1\right)}$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{8}{15n^9}\right)\cdot x^2-\frac{4}{5n^{10}}\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{4}{n^{11}}\right)\cdot x^4-\frac{22}{5n^{12}}\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{88}{5n^{13}}\right)\cdot x^6),$$

whence

ISSN 2959-6157

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}-\frac{(5n^2-10n+8)x^6}{240n^5}\right)$$
$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}+\frac{31\pi^2+8}{420n^7(4\pi^2+1)}$$
$$-\frac{1}{15n^8(4\pi^2+1)}-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{8}{15n^9}\right)\cdot x^2-\frac{4}{5n^{10}}\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{4}{n^{11}}\right)\cdot x^4-\frac{22}{5n^{12}}\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{88}{5n^{13}}\right)\cdot x^6)$$
$$\leqslant p_n(0,x)\leqslant$$
$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}\right)$$
$$\cdot exp(-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}-\frac{124\pi^2-1}{1680n^7(4\pi^2+1)}+\frac{1}{15n^8(4\pi^2+1)}-$$
$$\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{32\pi^2}{15n^9(4\pi^2+1)}\right)\cdot x^2+\left(\frac{4}{5n^{10}(4\pi^2+1)}\right)\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{16\pi^2}{n^{11}(4\pi^2+1)}\right)\cdot x^4+\left(\frac{22}{5n^{12}(4\pi^2+1)}\right)\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{352\pi^2}{5n^{13}(4\pi^2+1)}\right)\cdot x^6).$$

Using the Maclaurin series of the natural exponential function,

$$exp(-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}+\frac{31\pi^2+8}{420n^7(4\pi^2+1)}-\frac{1}{15n^8(4\pi^2+1)}$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{8}{15n^9}\right)\cdot x^2-\frac{4}{5n^{10}}\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{4}{n^{11}}\right)\cdot x^4-\frac{22}{5n^{12}}\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{88}{5n^{13}}\right)\cdot x^6)$$
$$\approx \quad 1-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}+\frac{31\pi^2+8}{420n^7(4\pi^2+1)}-\frac{1}{15n^8(4\pi^2+1)}$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{8}{15n^9}\right)\cdot x^2-\frac{4}{5n^{10}}\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{4}{n^{11}}\right)\cdot x^4-\frac{22}{5n^{12}}\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{88}{5n^{13}}\right)\cdot x^6,$$

$$exp(-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}-\frac{124\pi^2-1}{1680n^7(4\pi^2+1)}+\frac{1}{15n^8(4\pi^2+1)}$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{32\pi^2}{15n^9(4\pi^2+1)}\right)\cdot x^2+\left(\frac{4}{5n^{10}(4\pi^2+1)}\right)\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{16\pi^2}{n^{11}(4\pi^2+1)}\right)\cdot x^4+\left(\frac{22}{5n^{12}(4\pi^2+1)}\right)\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{352\pi^2}{5n^{13}(4\pi^2+1)}\right)\cdot x^6)$$
$$\approx \quad 1-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}-\frac{124\pi^2-1}{1680n^7(4\pi^2+1)}+\frac{1}{15n^8(4\pi^2+1)}$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{32\pi^2}{15n^9(4\pi^2+1)}\right)\cdot x^2+\left(\frac{4}{5n^{10}(4\pi^2+1)}\right)\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{16\pi^2}{n^{11}(4\pi^2+1)}\right)\cdot x^4+\left(\frac{22}{5n^{12}(4\pi^2+1)}\right)\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{352\pi^2}{5n^{13}(4\pi^2+1)}\right)\cdot x^6,$$

whence

$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}-\frac{(5n^2-10n+8)x^6}{240n^5}\right)$$
$$\cdot(1-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}+\frac{31\pi^2+8}{420n^7(4\pi^2+1)}-\frac{1}{15n^8(4\pi^2+1)}$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{8}{15n^9}\right)\cdot x^2-\frac{4}{5n^{10}}\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{4}{n^{11}}\right)\cdot x^4-\frac{22}{5n^{12}}\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{88}{5n^{13}}\right)\cdot x^6)$$
$$\leqslant p_n(0,x)\leqslant$$
$$\frac{1}{\sqrt{2\pi\left(n-\frac{x^2}{n}\right)}}\cdot\left(1-\frac{x^2}{2n}+\frac{(3n-2)x^4}{24n^3}\right)$$
$$\cdot(1-\frac{1}{4n}+\frac{1}{24n^3}+\frac{1}{180n^5}-\frac{124\pi^2-1}{1680n^7(4\pi^2+1)}+\frac{1}{15n^8(4\pi^2+1)}$$
$$-\left(\frac{1}{3n^3}-\frac{4}{15n^5}-\frac{2}{21n^7}+\frac{32\pi^2}{15n^9(4\pi^2+1)}\right)\cdot x^2+\left(\frac{4}{5n^{10}(4\pi^2+1)}\right)\cdot x^3$$
$$-\left(\frac{1}{3n^5}-\frac{2}{3n^7}-\frac{4}{9n^9}+\frac{16\pi^2}{n^{11}(4\pi^2+1)}\right)\cdot x^4+\left(\frac{22}{5n^{12}(4\pi^2+1)}\right)\cdot x^5$$
$$-\left(\frac{1}{3n^7}-\frac{56}{45n^9}-\frac{4}{3n^{11}}+\frac{352\pi^2}{5n^{13}(4\pi^2+1)}\right)\cdot x^6).$$

4. Conclusion

Now we have completed the estimates of the heat kernel on one-dimensional simple random walk. We reintroduced relevant notions through discussions in the field of graph theory and probability theory, and represented the factorial values through several common functions in the field of basic analysis. Finally, the results were obtained by expansions and calculations. The convergence rate of the heat kernel to 0 can be observed in the results, and it can be seen that the upper and lower bounds are close when $n \to \infty$. During the estimation process, due to the different number of terms retained during expansions, we provided three results with different degrees of accuracy that can be applied to different situations based on accuracy and complexity requirements. The method used in this paper is also possible to be expanded to higher dimensions, which we will explore in the future.

References

[1]A. Grigor'yan (2011-12). Analysis on Graphs. *Lecture Notes, University of Bielefield.*

[2] Ionescu Tulcea, C. T. (1949). Mesures dans les espaces produits. *Atti Accad. Naz.*

[3] *Lincei Rend*. 7: 208–211. Y. Li (2023). Basic Analysis (Vol. 2). *Lecture Notes, Southeast University.*