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Exploring Homotopy and Fundamental Groups: Definitions, Examples, and Key Propositions

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Abstract:

This article delves into key concepts in algebraic topology, specifically focusing on homotopy, contractible spaces, fundamental groups, and simply connected spaces. Definitions, examples, and key propositions of these concepts are explored, providing insights into their mathematical foundations and applications. Homotopy, describing the continuous deformation between two objects, plays a crucial role in defining homotopy groups, which serve as important invariants in algebraic topology. The fundamental group, as the first homotopy group, is instrumental in analyzing the basic shape of a topological space, capturing information about its structure, such as the presence of holes. The paper also examines contractible spaces, which can be continuously reduced to a single point, and simply connected spaces, characterized by their trivial fundamental groups. Additionally, the article addresses advanced topics in homotopy theory, including fibrations, exact sequences, and higher homotopy groups, highlighting their importance in linking topological spaces and revealing complex homotopy relationships. The study emphasizes the relevance of homotopy theory in both mathematics and broader fields, such as physics, where it aids in visualizing the universe's structure through analogies with fiber bundles.

Keywords: Algebraic Topology; Homotopy; Fundamental Groups; Simply Connected Spaces; Fibration.

1. Introduction

Algebraic topology is a fundamental branch of mathematics that studies topological spaces through algebraic methods. Among the core concepts of this field are homotopy and fundamental groups, which provide essential tools for understanding the structural properties of spaces, such as whether two spaces are homeomorphic [1]. Homotopy, which captures the idea of continuous deformation between objects, and fundamental groups, which reflect the basic shape and connectivity of spaces, are pivotal in analyzing topological questions. By examining these concepts, mathematicians can gain insights into the intrinsic properties of spaces that are not immediately apparent from their geometric descriptions.

Recent advancements in algebraic topology have seen an increased focus on homotopy and its applications to various mathematical and scientific problems. Homotopy theory has been used extensively in areas like group theory, complex analysis, and higher-dimensional topology, helping to build bridges between different mathematical disciplines [2]. The study of fundamental groups, as the first homotopy group, has also been crucial for identifying the characteristics of topological spaces, particularly in distinguishing between simply connected and multiply connected spaces [3]. Additionally, contractible spaces and simply connected spaces play vital roles in these analyses, offering simpler yet powerful models for understanding more complex structures.

This article explores the definitions, examples, and key propositions related to homotopy, contractible spaces, fundamental groups, and simply connected spaces. It provides a comprehensive overview of these concepts, including detailed explanations and practical examples to illustrate their significance in algebraic topology. Furthermore, the paper delves into advanced topics such as fibrations, exact sequences, and higher homotopy groups, highlighting their importance in linking topological spaces and revealing intricate homotopy relationships. By presenting these key ideas and demonstrating their applications, the study aims to provide a foundational understanding of these critical aspects of algebraic topology, with implications extending to broader fields, including physics and other sciences.The word topological space actually refers to a space and a topology of it, which implies the condition that a topology exists on this space. The formal definition of a topology can be given using open sets [2]. Consider a space X, and a collection of subsets $\mathfrak{F}$ of X that satisfies the following conditions:

X itself and $\varnothing$ are open;

The intersection of a finite number of open sets is open;

The union of any number of open sets is open.

Then $\mathfrak{F}$ is called the topology of space X, and $(X,\mathfrak{F})$ is called a topological space.

A group is an algebraic structure that satisfied the group axioms, consisting of a set and a binary operation. A group is a set $G(G\neq\varnothing)$ along with a binary operation [3]. The binary operation that denoted by " $\cdot$ " combines any two elements a and b to form an element in G, written as $a\cdot b$. The group axioms include 3 properties: associativity, identity element, and inverse element. Those properties will be explained in the following discussion of the validity of the fundamental group's definition, but they are only mentioned briefly here [4].

In calculus, a continuous function is defined by uses of ? – δ language. That is, a function f is called continuous at point $x_0\in\mathbb{R}^n$, if $\forall ?>0$, $\exists\delta>0$ such that $\forall x\in\mathbb{R}^n$, if $\left|\,|x-x_0|\,\right|<\delta$, then $\left|\,|f(x)-f(x_0)|\right|<?$. Since an open ball describe the set of all points that have distance from the centre less than r, a continuous function can be defined using open balls [5]. Consider a function f describes a continuous mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, $\forall$ open set $U\in\mathbb{R}^m$, $\forall x_0\in f^{-1}(U)$, $f(x_0)\in U$, $\exists ?>0$ such that $B(f(x_0);?)\subset U$; $\exists\delta>0$ such that $f(B(x_0;\delta))\subset B(f(x_0);?)$. So $B(x_0;\delta)\subset f^{-1}(B(f(x_0);?))\subset f^{-1}(U)$, meaning that $f^{-1}(U)$ is an open set in $\mathbb{R}^n$. Therefore, it can be concluded that f is a continuous function if the preimage of all open sets under the function is open [6].

With the fundamental concepts above, explaining the definitions of homotopy and fundamental groups will be easier.

2. Homotopy

2.1 Definition 1: Introduction and Formal Definition of Homotopy

The definition of homotopy captures the concept of two paths being the same in a topological sense. If two paths can be smoothly deformed into one another, they are said to be homotopic [7]. The following is the definition of homotopy:

Let $c,d:[0,1]\to X$ be two paths of a topological space X from x to y. We say that c and d are homotopic if there exists a continuous function $F:[0,1]\times[0,1]\to X$ such that $c(t)=F(t,0)$ and $d(t)=F(t,1)$ for all $t\in[0,1]$. The function F is called an homotopy.

In this definition, two paths c,d are connected through a continuous function F, which makes them can be continuously transformed into one another. This is called homotopy, denoted as $c\simeq d$.

2.2 Example 1: Practical Example Illustrating the Concept of Homotopy

For the definition of homotopy above, this paper gives

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several examples to explain.

For the first example, as Fig. 1 shows, consider the two paths as two dashed lines on a plane with the same endpoints. If the two dashed lines are homotopic relative to their endpoints, then all curves with the same endpoints as the existing ones inside the closed shape formed by these two curves represent all possible homotopies [8]. It is important to note that the curves mentioned here do not intersect themselves [9]. If we take the second parameter of the homotopy F, whose value range is $[0,1]$, as time, it can be seen as an animation describing the continuous deformation between the two curves.

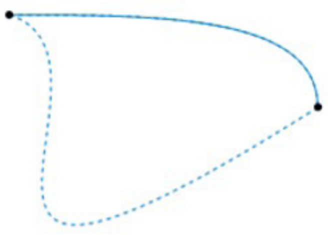

Fig. 1 A possible homotopy between two dashed lines (Photo credit: Original).

The second example is about a linear homotopy on a 2-dimentional Euclidean space $\mathbb{R}^2$. Consider two topological spaces $X=[0,1]$ and $Y=R$, let the first path be $f(x)=2x$, the second path be $g(x)=0$, then the homotopy F between them can be defined as $F(x,t)=(1-t)f(x)+t\cdot g(x)=(1-t)\cdot 2x$. Obviously, when $t=0$, $F(x,0)=2x=f(x)$, while $F(x,1)=0=g(x)$ when $t=1$ [10]. This shows that function f can be continuously transformed into the function g via the homotopy F.

3. Contractible Spaces

3.1 Definition 2: Definition and Characteristics of Contractible Spaces

The definition of contractible spaces can be given by using homotopy. A contractible space is also an important concept in topology because it describes a space that can be continuously shrunk to a single point. This implies that a contractible space must have a trivial topology, as a topological space with a trivial topology is one where the only open sets are the empty set and the entire space. The following is the definition of contractible spaces:

Consider a homotopy between two continuous functions $f,g:X\to Y$ from a topological space X to another one Y. In the case, $X=Y$, $f=id_x$ and g is constant, we say that X is contractible.

In the definition above, id_x refers to the identity map, which is a mapping of X to itself. The meaning of the definition is that if there exists a homotopy showing that the identity map of a topological space X is null-homotopic, which means be homotopic to a constant map, this implies that the topological space X can be continuously shrunk to a point, meaning X is a contractible space.

3.2 Example 2: Illustration of a Contractible Space Through an Example

A positive and a negative example can be used to explain the definition above.

First, consider the example of a star-shaped subspace in an $\mathbb{R}$-vector space X. For X, consider if there exists a star-shaped subspace at x as S_x, $\forall y\in S$, $(1-t)x+ty\in S$, $\forall t\in[0,1]$. The identity map $id_{S_x}:S_x\to S$ is homotopic to the constant map that sends every point in S to the point x. The homotopy can be defined as $F:[0,1]\times S_x\to S_x$, $(t,y)\mapsto(1-t)y+tx$. Since S_x can be continuously shrunk to a point within itself, it is therefore contractible.

Next is a counterexample involving an annular space E. As shown in Fig. 2, E can be regarded as a 1-dimensional topological space that is homeomorphic to the circle S^1. Assume E is contractible. According to the definition of contractibility, there must exist a homotopy $H:E\times[0,1]\to E$ such that:

$$\forall x\in E, H(x,0)=x \text{ (the identity map)} \tag{1}$$

$$\forall x\in E, H(x,1)=x_0 \tag{2}$$

Where x_0 is a fixed point in E. To contract E to the point x_0, points on opposite sides of the boundary of E must approach each other and eventually coincide at x_0. However, it is impossible to perform such a continuous deformation within E because E has no interior region. Therefore, these points cannot cross the space's interior while maintaining the continuity required by the homotopy, violating the requirement that the homotopy remains continuous throughout the process. Thus the assumed homotopy H is not continuous, meaning E is not contractible.

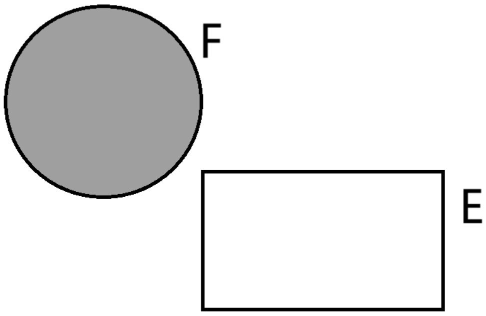

Fig. 2 Space E and F (Photo credit: Original).

4. Fundamental Groups

4.1 Definition 3: Detailed Explanation of Fundamental Groups

Another application of homotopy is in homotopy groups, with the fundamental group being the first and simplest homotopy group. By analyzing the fundamental group of a topological space, its basic shape can be understood. Therefore, the fundamental group is an important concept for determining whether two spaces are homeomorphic. Its definition is as follow:

Let X be a topological space and $x_0 \in X$. The set of equivalent classes of closed paths from x_0 to x_0 under homotopy equivalence is denoted by $\pi_1(x_0, X)$. This set has a group structure and is called the fundamental group at the base point x_0.

There is also a key proposition that related to the definition, which is:

$\pi_1(x_0, X)$ only depends up to isomorphism of the path-connected component of x_0.

The proof of this proposition will be explained in Section 4.3.

4.2 Example 3: Example Demonstrating the Application of the Fundamental

For the definition of the fundamental group, an intuitive explanation is provided using the example of a 2-dimensional torus. As shown in Fig. 3, consider a fixed point p on the 2-dimensional torus T^2. After fixing the point, a closed curve γ_1 can be constructed starting and ending at this point. The homotopy class of this curve, denoted $\pi(p, T^2)$, represents the fundamental group of the torus based at the point p. The homotopy class of γ_1 includes all curves homotopically equivalent to it. Imagine γ_1 as being freely deformable and infinitely extendable without breaking; then the new curve obtained by deforming γ_1 is one that is homotopically equivalent to γ_1. As a counter-example, γ_1 and γ_2 are not homotopically equivalent in the figure. γ_1 loops around the hole in the middle of the torus, while γ_2 loops around the hollow part of the ring, since T^2 is formed by rotating a hollow circle.

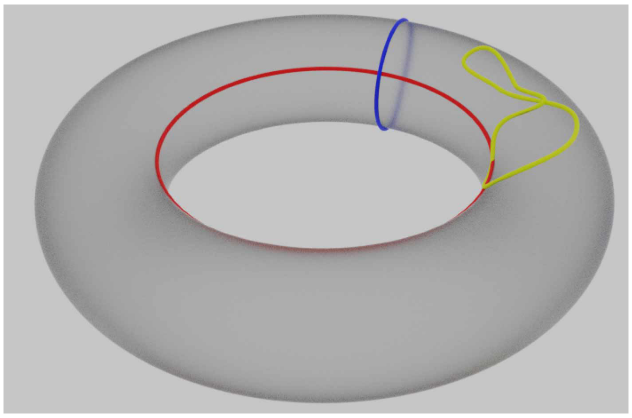

Fig. 3 Loops on the 2-dimentional torus (Photo credit: Original).

4.3 Proof of the Proposition

Let $\Omega(X, x_0)$ be the set of all loops in X based at x_0. $r \in \Omega(X, x_0)$ is a continuous map $r:[0,1] \to X$ such that $r(0) = r(1) = x_0$. Define a binary operation $*$ on $\pi_1(x_0, X)$ such that the concatenation of two loops r_1 and r_2 in $\Omega(X, x_0)$, $r_1 * r_2$, is the loop defined by:

$$r_1 * r_2(t) = \begin{cases} r_1(2t), if\, 0 \leqslant t \leqslant \frac{1}{2}, \\ r_2(2t-1), if\, \frac{1}{2} \leqslant t \leqslant 1. \end{cases} \quad (3)$$

For any path p from x_0 to x_1, there exists the map $[r] \mapsto [p * r * p^{-1}]$ such that p can induce an isomorphism between the fundamental groups, where $p^{-1}(t) = p(1-t)$ is the path from x_1 back to x_0 ($[r]$ means the homotopy class of r). Thus, the fundamental group $\pi_1(x_0, X)$ is isomorphic to $\pi_1(x_1, X)$ for any other base point in X.

4.4 Verification of Group Axioms

To prove that $\pi_1(x_0, X)$ forms a group under operation $*$,

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it is needed to check group axioms, which are associativity, identity element and inverse element.
First, the operation $*$ is associative up to homotopy. That is, for any three loops r_1, r_2, r_3, the loops $(r_1 * r_2) * r_3$ and $r_1 * (r_2 * r_3)$ are homotopic. Hence, the induced operation on $\pi_1(x_0, X)$ is associative.
The constant loop $e(t) = x_0$ for all $t \in [0,1]$ serves as the identity element. For any loop r, the concatenation $e * r$ and $r * e$ are both homotopic to r. Hence, the homotopy class of e is the identity element in $\pi_1(x_0, X)$.
For each loop r, define its inverse loop r^{-1} by $r^{-1}(t) = r(1-t)$. The concatenation $r * r^{-1}$ is homotopic to the constant loop e. Hence, $[r]^{-1} = [r^{-1}]$ serves as the inverse of $[r]$ in $\pi_1(x_0, X)$.

5. 5 Simply Connected Spaces

5.1 Definition 4: Definition and Properties of Simply Connected Spaces

The concept of simple connectivity is very important in complex analysis, and it is a property of topological spaces in topology. Using the concepts of fundamental groups and trivial groups, the definition of a simply connected space can be given as follows:
If X is path connected and its fundamental group is trivial, we say that X is simply connected.
A trivial group refers to a group that contains only a single element e, with the group operation being $e + e = e$.

5.2 Example 4: Example to Elucidate the Concept of Simply Connected Spaces

For a simply connected space, a counterexample is used to illustrate its properties. As shown in Fig. 4, there is a set with a yellow closed curve inside. Since this closed curve encloses two holes in the set, in cannot be contracted to a single point, meaning it is not homotopically equivalent to a point. So, the set is not a simply connected space.

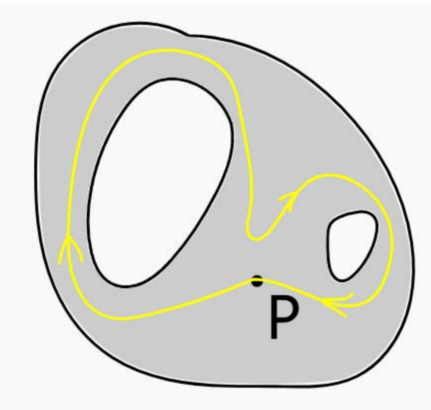

Fig. 4 A set with a yellow closed curve inside (Photo credit: Original).

Additionally, an obvious example is the 2-dimensional Euclidean space $\mathbb{R}^2$. $\mathbb{R}^2$ is path-connected, and all loops in $\mathbb{R}^2$ can be contracted to a single point, meaning its fundamental group is trivial. Therefore, $\mathbb{R}^2$ is a simply connected space.

5.3 Key Proposition of Simple Connection

There's a crucial proposition that related to simple connection, which connect it with properties of contractible:
If space X is contractible, then it is simply connected.
Here's the proof of the proposition:
First, according to the definition of simple connection, it is needed to prove that X is path-connected. Since X is contractible, for any $x_1, x_2 \in X$, the homotopy H provides a path to move from x_1 to the constant point c and then from c to x_2. Thus, any two points in X can be connected by a path.
Next, if the fundamental group of X is trivial, X is simply connected is obvious. Let $r[0,1] \to X$ be any loop in X based at a point $x_0 \in X$. Let F be the homotopy such that $F(x,0) = x$, and $F(x,1) = c$ for all $x \in X$. Consider the homotopy $H:[0,1]\times[0,1] \to X$ defined by: $H(s,t) = F(r(s),t)$. Inside, $H(s,0) = r(s)$, which is the original loop; $H(s,1) = c$, which is the constant map to the point c. For $s = 0$ and $s = 1$, $H(0,t) = H(1,t) = c$, which is fixed at the point c. Hence, r is homotopic to the constant loop, meaning that in the fundamental group $\pi_1(X)$, the loop r represents the identity element. Since any loop in X can be homotopically contracted to a point, the fundamental group $\pi_1(X)$ is trivial.

6. Advanced Topics in Homotopy Theory

In the study of homotopy theory, many advanced topics revolve around fibration, exact sequences, and higher homotopy groups.

7. Fibration and Its Properties

7.1 The Definition of Right Lifting Property, Fibration, and Pullback Fibration

Fibration is a key concept that not only involves the structure of topological spaces but also provides essential tools for understanding complex homotopy relationships. A fibration connects different topological spaces through a special type of continuous map and serves as a bridge in homotopy theory. This type of mapping satisfies the right lifting property for any space X. It is defined by a continuous map $f: X \to Y$. F has the right lifting property with respect to continuous map $A \to B$, if for any commutative diagram of continuous maps:

$$\begin{array}{c} A \to X \\ \downarrow\downarrow f \\ B \to Y \end{array} \tag{4}$$

There exists a continuous map $B \to X$ such that the diagram:

$$\begin{array}{c} A \to X \\ \downarrow\downarrow f \\ B \to Y \end{array} \tag{5}$$

$$\begin{array}{c} A \to X \\ \downarrow \nearrow \downarrow f \\ B \to Y \end{array} \tag{6}$$

is commutative. Thus, a fibration is a continuous map that satisfies the right lifting property with respect to the inclusion $A \to A \times [0,1], a \mapsto (a,0)$ for any topological space A.

The properties of fibration are crucial for constructing and analyzing the homotopy types of topological spaces. When studying fibrations, particular attention is paid to the pullback property of fibrations, the homotopy type of fibers, and the applications of fibrations in homology and homotopy groups. Through these properties, new topological spaces that are homotopy equivalent to the original space can be constructed, leading to a deeper understanding of the homotopy relationships between sapces.

For instance, consider the pullback property of a fibration. Given a fibration and a map, $p: E \to B, f: A \to B$, the map $p_f : f^*(E) \to A$ is a fibration, and $f^*(E) = \{(a,e) \in A \times E \mid f(a) = p(e)\}$ is the pullback, and the projections of $f^*(E)$ onto A and E yield the following commutative diagram:

$$\begin{array}{c} f^*(E) \to E \\ Pf \downarrow\downarrow P \\ A \xrightarrow{f} B \end{array} \tag{7}$$

Then, the fibration p_f is called the pullback fibration.

7.2 Examples of Pullback Fibration

A related intuitive example can be given. Imagine A as a line, with a circle (fiber) at each point on A.This means the total space E looks like a series of circles stacked on A.

Now, imagine a more complex shape B, e.g., a zigzag curve, and the map f projects points on B onto the line A. When the fibration is pulled back from A to B, the same circles (fibers) corresponding to the points on A are effectively attached to B, and the difference is that these circles now cover the more complex shape of B.

As a result, the pullback fibration creates a new structure that combines the original fibers with the geometry of B, while preserving the relationship determined by f.

8. Exact Sequences and Higher Homotopy Groups

8.1 The Definition of Exact Sequences

In the context of fibration, the exact sequence provides a powerful tool for analyzing the complex structure of homotopy groups. The long exact sequence of a fibration connects higher-order homotopy groups to lower-order ones. These sequences not only help researchers understand the homotopy groups of individual spaces but also reveal the homotopy relationships between different spaces.

The definition of exact sequences comes from group theory. A sequence

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} G_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} G_n \tag{8}$$

of groups and group homomorphisms will be exact at G_m, if $im(f_m) = ker(f_{m+1})$. Ther sequence will be an exact sequence if the sequence is exact for all G_m, which $m \in [1,n)$ and is an integer.

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8.2 Theorem Related to Exact Sequences

After introducing the definition of exact sequences, here is a crucial theorem related to exact sequences:

Let $f: E \to B$ be a fibration, $e_0 \in E$, $b_0 ? f(e_0)$, F is the fiber above b_0, then there exists a long exact sequence:

$$\cdots \to \pi_1(e_0, F) \to \pi_1(e_0, E) \to \pi_1(e_0, B) \xrightarrow{\delta} \pi_0(e_0, F) \to \pi_0(e_0, E) \to \pi_0(e_0, B) \quad (9)$$

Note that the exactness at π_0 holds for those points whose image is connected to the base point, this concept is analogous to the concept of kernel in the group.

8.3 Higher Homotopy Groups

Higher homotopy groups are an important subject of study in homotopy theory, capturing deeper topological information about a space. The computation of higher homotopy groups often relies on tools like fibrations and exact sequences. These tools allow researchers to progressively decompose and analyze the homotopy type of a space and use known homotopy group information to deduce higher-order homotopy groups. For the n-th homotopy group $\pi_n(X)$, it describes the homotopy classes of maps from the n-dimensional sphere S^n to the space X. For example, the second homotopy group $\pi_2(X)$ is related to 2-dinemsional "holes" in the space, as Fig. 5 shown.

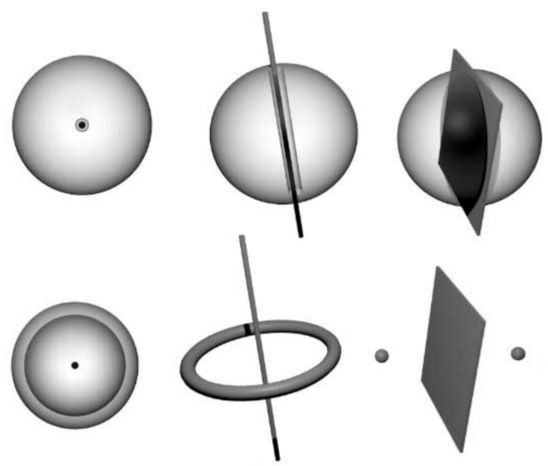

Fig. 5 The three types of holes in 0, 1, and 2-dimensions (Photo credit: Original).

9. Conclusion

This paper provides a comprehensive overview of key concepts in algebraic topology, focusing on homotopy, contractible spaces, fundamental groups, and simply connected spaces. By defining these fundamental ideas and illustrating them with practical examples, the study highlights the critical role of homotopy in understanding the intrinsic properties of topological spaces. The exploration extends to advanced topics such as fibrations, exact sequences, and higher homotopy groups, emphasizing their significance in linking topological spaces and unraveling complex homotopy relationships. These insights not only deepen the understanding of algebraic topology but also demonstrate its broad applications in various fields, including physics, where analogous structures aid in visualizing complex phenomena like the shape of the universe. Future research in this area could focus on further exploring the connections between homotopy theory and other mathematical disciplines, such as category theory and differential geometry, to uncover deeper structural insights. Additionally, there is significant potential in applying advanced homotopy concepts, like higher homotopy groups and fibrations, to emerging fields such as data science, machine learning, and quantum computing, where topological methods can provide novel ways to analyze data structures and solve complex problems. Expanding the theoretical framework of homotopy to include these modern applications could bridge gaps between abstract mathematical theory and practical, real-world challenges, paving the way for new innovations across diverse scientific domains.

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