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The Applications of Derivative and Differential

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Abstract:

Differential and derivative are core concepts in analytical mathematics and are closely related. To be brief, the derivative refers to the rate of change of a function near a point, while the differential describes the actual change in the value of the function under the action of the derivative. Together, they form the basis of calculus. Differential is an application of the concept of derivative, specifically the response to a small change in the function. The derivative is the slope of the local linear approximation, while the differential is the increment of the function value under this linear approximation. This paper mainly discusses the topic about derivative and differential. To illustrate them clearly, this paper will introduce step by step, from their concept to examples, including definition, properties and geometric meaning. Specifically, the article particularly emphasizes applications in sequence problems and approximate calculations. These applications are of great significance in mathematics, physics, engineering, and economics.

Keywords: Derivative; differential; applications.

1. Introduction

The study of derivative and differential originated from Newton and Leibniz's description of instantaneous rate of change. It was strictly defined by Cauchy and other mathematicians in the 19th century, laying the theoretical foundation for calculus. In short, the derivative is a certain value, which means the slope of the tangent line in geometry. The differential is a function expression used to calculate the approximate value of the dependent variable when the independent variable changes slightly.

Derivative and differential have always been hot topics in numerous domains. In 2000, Tian's paper gave the specific application of derivatives in the method of finding the limit of a series and the approximate solution of a type of equation [1]. In 2003, Zheng gave the approximate calculation formula of logarithmic differential and compared with the classical approximate calculation formula [2]. In 2011, Tang provided a convenient and effective tool using derivative to analyze and solve some function problems, leading to a new perspective and new method [3]. In 2013, Cao used calculus to analyze physical problems [4]. In 2018, Liu, Wang and Zhang did the research on the application of differential quadrature method in engineering structural dynamics [5]. In 2021, Wang used derivatives to study the problem of finding function parameters [6]. In 2022, Darmayanti, Baiduri and Sugianto gave the introduction of learning application derivative algebraic functions [7]. In 2023, Tunç concerned about fractional derivative

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delay integral-differential equations [8]. However, there's little discussion about applications in sequence problems and approximate calculations. Hence, this paper is going to supplement some contents of this area and offers some advice on solving similar problems.

This paper is organized as follows. In section 2.1, basic information of derivative is given. In section 2.2, applications of derivatives in proving inequalities are given. In section 2.3, applications of derivatives in sequence problems are given. In section 2.4, introduction of differential is given. In section 2.5, applications in differential approximate calculations are given.

2. Derivative and Differential

2.1 Basic Information of Derivative

General definition: The derivative is a local property of a function, describing the rate of change of a function near a certain point. If the independent variable and value of the function are real numbers, the derivative of the function at a certain point is the slope of the tangent line of the curve represented by the function at this point. The essence of the derivative is to make a local linear approximation of the function through the concept of limit.

Rules for some basic functions:

$$\frac{d}{dx}x^a = ax^{a-1}\ \frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}lnx = \frac{1}{x}\ \frac{d}{dx}\sin x = cosx\ \frac{d}{dx}cosx = -\sin x \quad (1)$$

Geometric meaning: The derivative is regarded as the slope of a tangent line, making it an important tool for understanding the local behavior of a function. When the derivative of a function is positive, it can be inferred that the function is rising in this area; if the derivative is negative, the function is falling; if the derivative is zero, it may be a local maximum or minimum point of the function.

The relationship between derivatives and differentials: Differentials and derivatives are both used to describe the rate of change of a function. Differentials are the rate of change of a function at a certain point, while derivatives describe the rate of change of the entire function at a certain point. Differentials can be seen as the local linear approximation of a function at a certain point, while derivatives can be viewed as the global rate of change of a function in the entire domain of definition. It can be understood that differentials are local approximations of derivatives, while derivatives are the overall summary of differentials.

2.2 Applications of Derivatives in Proving Inequalities

Example 1: [9] Prove that the inequality holds $ln(x+1) > x - \frac{x^2}{2}$, where $x > 0$.

Proof: Suppose that $f(x) = ln(x+1) - x + \frac{x^2}{2}$, then $f'(x) = \frac{1}{1+x} + x - 1 = \frac{x^2}{1+x} > 0$.

So, $f(x)$ is monotonically increasing in the domain of x, then $f(x) > f(0) = 0$.

So, $ln(x+1) - x + \frac{x^2}{2} > 0$, thus $ln(x+1) > x - \frac{x^2}{2}$.

Example 2: Prove that the inequality holds $4xlnx > x^2 + 2x - 4$, where $0 < x < 2$.

Proof: Suppose that $f(x) = 4xlnx - x^2 - 2x + 4$, then $f'(x) = (4 + 4lnx) - 2x - 2$.

Let $f'(x)$ =0, $x = 1$; $f''(1) = 2 > 0, x = 1$ is the minimum point of $f(x)$.

$f''(x) > 0$, so $f(x) > f(1) = 1 > 0$. $4xlnx - x^2 - 2x + 4 > 0$.

Thus, $4xlnx > x^2 + 2x - 4$.

2.3 Applications of Derivatives in Sequence Problems

Example 3: [10] Find the sum S_n of the first n terms of the sequence $\{nx^{n-1}\}$, where $n \in N^*$.

Proof: when $x = 1$, $S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

when $x \neq 1$, $\because x + x^2 + x^3 + \cdots + x^n = \frac{x - x^{n+1}}{1-x}$.

Take the derivative of both sides:

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2} \quad (2)$$

Overall, when $x = 1$, $S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$; when $x \neq 1$, $S_n = \frac{1-(n+1)x^n + nx^{n+1}}{(1-x)^2}$.

Example 4: Given that the general term of the sequence $\{a_n\}$ is $a_n = 8n^2 - n^3$, where $n \in N^*$, find the largest term of $\{a_n\}$.

Proof: Suppose that $f(x) = 8x^2 - x^3 (x > 0)$, then $f'(x)=16x-3x^2$.

When $0<x<\frac{16}{3}$, $f'(x)>0$; when $x>\frac{16}{3}$, $f'(x)<0$.

When $x=\frac{16}{3}$, the function reaches its maximum value.

Because $n\in N^*$, it can be inferred that $f(5)=75, f(6)=72, 75>72$.

So, the largest term of $\{a_n\}$ is 75.

2.4 Introduction of Differential

Basic definition: If the function y=f(x) has a derivative f'(x) at point x, then the change in y caused by the change in x(Δx) is $\Delta y=f(x+\Delta x)-f(x)=f'(x)\cdot\vartriangle x+o(\vartriangle x)$, where $o(\vartriangle x)$ tends to 0 as $\vartriangle x$ approaches 0. Therefore, the main part of the linear form of Δy: $dy=f'(x)\bullet\vartriangle x$ is the differential of y.

Overall, in calculus, the differential of a function is a linear description of the local changes of a function. Differentials can approximately describe how the value of a function changes when the value of the function's independent variable changes sufficiently.

Some important properties:

(1) Chain Rule: $dy=\frac{dy}{du}\bullet\frac{du}{dx}\bullet dx$

(2) Product Rule: $d(uv)=vd(u)+ud(v)$

Application 1: Consider the ideal gas law $pV=nRT$. Relate the pressure p, volume V, and temperature T of an ideal gas, where R is a physical constant, and n is the number of moles of gas [11].

$$d(pV)=pdV+Vdp=nRdT \tag{3}$$

Linearity: $d(au+bv)=ad(u)+bd(v)$, when a,b are constants. df=dfdxdx $d(f(x))=f'(x)\bullet dx.$

Application 2: Simplify $d(3x+2x^2)$

Proof:

$d(3x+2x^2)=3dx+2d(x^2)=3dx+4xdx=(4x+3)dx.$

Further exploration: Find the derivative of $f(x)=ln(\sin x)$.

First, set $u=\sin x$, so that $d(f(x))=d(ln(u))=\frac{1}{u}\bullet du$ $=$ $\frac{1}{\sin x}\bullet cosx\bullet d(x)=cotx\bullet d(x).$

Geometric meaning: Let Δx be the increment of the point on the curve $y=f(x)$ on the horizontal axis, Δy be the increment of the curve on the vertical axis corresponding to Δx at point P, and dy be the increment of the tangent line of the curve on the vertical axis corresponding to Δx at point P. When Δx is very small, $|\Delta y-dy|$ is much smaller than $|\Delta y|$, so near point P, the tangent line segment can be used to approximate the curve segment.

When the independent variable increases from x_0 to $x_0+\Delta x$, the function increment $\Delta y=f(x_0+\Delta x)-f(x_0)$ is the increment corresponding to Δx on the tangent line at point P:

$$dy=f'(x_0)\bullet\Delta x \tag{4}$$

2.5 Applications in Differential Approximate Calculations

Differential has plenty of applications in mathematics. This part aims to introduce some applications in approximate calculations.

According to the relation of function increment and differential:

$$\Delta y=f'(x0)\cdot\vartriangle x+o(\vartriangle x)=dy+o(\vartriangle x) \tag{5}$$

When $x\approx x_0$

$$f(x)\approx f(x_0)+f'(x_0)(x-x_0) \tag{6}$$

Example 5: Find the approximate value of $\sin 63°$

Proof: Because $\sin 63°=\sin(\frac{\pi}{3}+\frac{\pi}{60})$.

Then, set $f(x)=\sin x, x_0=\frac{\pi}{3}$, $\Delta x=\frac{\pi}{60}$.

$Hence, \sin 63°\approx\sin\frac{\pi}{3}+cos\frac{\pi}{3}\bullet\frac{\pi}{60}=\frac{\sqrt{3}}{2}+\frac{1}{2}\bullet\frac{\pi}{60}\approx 0.918$.

Example 6: [10] Suppose that the period T of a single pendulum is $1s$, and the pendulum length can be shortened by at most $0.01cm$, how many seconds per day can this pendulum be faster?

Proof: From physics, the relationship between the period T of a single pendulum and the length l of the pendulum is:

$$T=2\pi\sqrt{\frac{l}{g}} \tag{7}$$

Where g is the acceleration due to gravity, and the period of the single pendulum is $0.1s$, then the original length of pendulum is:

$$l_0=\frac{g}{4\pi^2} \tag{8}$$

So $\Delta T\approx\frac{dT}{dl_0}\cdot\Delta l=\frac{\pi}{\sqrt{g}}\cdot\frac{1}{\sqrt{l_0}}\Delta l\approx -0.0002(s)$

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Then it can be faster: $60 \cdot 60 \cdot 24 \cdot 0.0002 = 17.28s$.

3. Conclusion

Overall, differential and derivative are used widely in daily lives. They are powerful tools to reduce artificial work accounts. Speaking of the application of derivative in this paper, its use in sequence problems breaks away from traditional methods and provides new solutions to such problems. The conventional method is to merely consider sequence solutions while it usually consumes more time. Using derivative saves lots of unnecessary process in large extent. Differential can estimate the change in the value of a function, which is usually very accurate, especially over a small range. Actually, it can be applied to other subjects. For example, in physics, differential can be used to estimate the speed and acceleration of an object, by analyzing the object‘s state of motion. In economics, differentials are used to estimate the tiny changing trend of costs to help decision makers make better choices. It plays a vital role in assisting investors to acquire remarkable benefits. Moreover, differential approximation can also be used for error analysis, helping us evaluate the accuracy and stability of calculation results. In engineering and experimental design, differential approximation can estimate the impact of small changes in input variables on output results, perform sensitivity analysis and optimize design. Making mistakes in this aspect can be deadly, while with the help of it, the worries are no need to concern. In addition, the drawback is that this paper only provides rough ideas and lacks of detailed applications and instances in wider fields. Nonetheless, this paper hopes to provide readers with more innovative approaches to solving such problems and more creative thinking to other areas.

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