Dean&Francis

ISSN 2959-6157

The 2-Sheeted, 3-Sheeted, and Universal Coverings of Corresponding 2-Oriented Graph of Rank-2 Free Group

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Abstract:

This paper reviews results from classical covering theory and applies the classification theorem to provide a comprehensive classification of the connected 2-sheeted and 3-sheeted covering spaces of the bouquet of 2 circles, up to isomorphism without basepoints. These coverings are represented as 4-valent graphs, labeled by the two generators of the rank-2 free group, which serves as the fundamental group of the bouquet of 2 circles. The result directly addresses a question posed by Hatcher, a problem often answered incompletely in previous attempts. This paper provides all possible cases and includes detailed illustrations to accompany the classifications. Additionally, the paper explores the X-digraphs associated with the corresponding index-2 and index-3 subgroups of the rank-2 free group. These subgroup graphs introduce powerful combinatorial tools to examine the subgroups of free groups. In the broader context of geometric group theory, the paper concludes by discussing the Cayley graph of the rank-2 free group as a universal covering space.

Keywords: Covering; free group; Cayley graph; geometric group theory.

1. Introduction

Graphs have been a central tool in algebra and topology, providing powerful ways to visualize and solve abstract problems. The idea of representing groups using graphs, particularly Cayley graphs, originated with Arthur Cayley in the 19th century. By associating group elements with vertices and generators with edges, Cayley graphs enable a combinatorial representation of groups. This technique has since become a standard method for tackling algebraic problems through graphical and combinatorial approaches.

In algebraic topology, covering spaces and group cohomology are deeply linked through the framework of Eilenberg-MacLane spaces, denoted $K(G,1)$. These spaces play a central role because they offer a homotopy-theoretic interpretation of cohomology [1, 2]. Specifically, for a space with fundamental group G, a $K(G,1)$-space is characterized by having G as its only nontrivial homotopy group, while all higher homotopy groups vanish.

In this context, covering spaces are closely related to cohomological extensions of groups. For instance, an

ISSN 2959-6157

extension of a group G by an abelian group A corresponds to a cohomology class in $H^2(G,A)$. The structure of the covering space reflects how the fundamental group G acts on these spaces, and classifying covering spaces can often be framed as a cohomological problem. This interaction between homotopy, cohomology, and covering spaces is crucial for understanding how spaces with complicated fundamental groups decompose into simpler parts, particularly through their subgroups and quotient spaces.

This paper leverages some of these ideas to classify, up to isomorphism of covering spaces without basepoints, the connected 2-sheeted and 3-sheeted covering spaces of the bouquet of 2 circles $S^1 \vee S^1$, a fundamental example of a $K(F_2,1)$-space, where F_2 is the free group of rank 2. This answers a question from [3]. To the knowledge, most of the answers to the question are incomplete due to the elementary and enumerative nature of the finding. The paper also explores the associated subgroup graph, which are X-digraphs that trace back to Bass-Serre theory and are canonically identified with orbit space of action on subtree [4]. The last section discusses the universal covering of $S^1 \vee S^1$ in a geometric-group-theoretic framework.

2. Basic Theory in Topology and Group

Basic notations and theories are listed in this section for later computation. In the context of covering, the base space is $X = S^1 \vee S^1$, wedge sum of two circles. It is also termed with bouquet of two circles. It is a 2-oritented graph. The fundamental group of X is $G = \pi_1\left(S^1 \vee S^1\right) \cong F_2$. The Cayley graph of a group, which serves as the universal covering of X, is defined below.

Definition 1 (Cayley graph): For group G and generating set $S \subseteq G$, the Cayley graph of G with generating set S, denoted by $Cay(G,S)$, is the graph (V,E) where: Vertex set V is all of the elements of G; edge set E is defined by:

$$E = \{\{g, g\cdot s\} \mid g \in G, s \in (S \cup S^{-1}) \setminus \{e\}\} \tag{1}$$

2.1 Covering Theory

Some results from Hatcher will be need [3].

Proposition 1: The number of sheets of a covering space $p:\left(X, x_0\right) \to \left(X, x_0\right)$ with X and X path-connected equals the index of $p_*\left(\pi_1\left(X, x_0\right)\right)$ in $\pi_1\left(X, x_0\right)$ where $p_*:\pi_1\left(X, x_0\right) \to \pi_1\left(X, x_0\right); [\gamma] \mapsto [p \circ \gamma]$ is the induced homomorphism [3].

Proposition 2: Suppose X is path-connected, locally path-connected, and semilocally simply-connected. Then for every subgroup $H \subset \pi_1\left(X, x_0\right)$ there is a covering space $p: X_H \to X$ such that $p_*\left(\pi_1\left(X_H, x_0\right)\right) = H$ for a suitably chosen basepoint $x_0 \in X_H$ [3].

Theorem 1 (Classification Theorem): Let X be path-connected, locally path-connected, and semilocally simply-connected. Then there is a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces $p:\left(X, x_0\right) \to \left(X, x_0\right)$ and the set of subgroups of $\pi_1\left(X, x_0\right)$, obtained by associating the subgroup $p_*\left(\pi_1\left(X, x_0\right)\right)$ to the covering space $\left(X, x_0\right)$. If basepoints are ignored, this correspondence gives a bijection between isomorphism classes of path-connected covering spaces $p: X \to X$ and conjugacy classes of subgroups of $\pi_1\left(X, x_0\right)$ [3].

Proposition 3: Let $p:\left(X, x_0\right) \to \left(X, x_0\right)$ be a path-connected covering space of the path-connected, locally path-connected space X, and let H be the subgroup $p_*\left(\pi_1\left(X, x_0\right)\right) \subseteq \pi_1\left(X, x_0\right)$. Then this covering space is normal if and only if H is a normal subgroup of $\pi_1\left(X, x_0\right)$ [3].

2.2 Group Theory

A theorem on counting the number of index-k subgroup of free group F_r of rank r due to Hall will be needed to confirm the enumerative method this paper used [5].

Theorem 2: The number $N(k,r)$ of subgroups of index k in rank-r free group F_r is given recursively by:

$$N(k,r) = k(k!)^{r-1} - \sum_{i=1}^{k-1}\left((k-i)!\right)^{r-1} N(i,r), \forall k \geqslant 2 \tag{2}$$

where $N(1,r) = 1$ [5].

3. Connected 2-Sheeted and 3-Sheeted Coverings of $S^1 \vee S^1$

3.1 Connected 2-Sheeted Coverings of $S^1 \vee S^1$

This section finds all connected 2-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphism of covering spaces without basepoints.

First note that $X = S^1 \vee S^1$ is path connected and locally path-connected and the latter property is also inherited by the covering space X, so X is connected if and only if it is path-connected. By proposition 1, each 2-sheeted cover $p: X \to X$ has its induced subgroup $H = p_*\left(\pi_1\left(X\right)\right)$ with index 2 in $\pi_1(X) = \pi_1(S^1 \vee S^1) = ?a?*?b?$. Thus, by the classification theorem, classifying connected 2-sheeted covering spaces up to isomorphism is equivalently of finding all index-2 subgroups of $F_2 = ?a?*?b?$. Since every index-2 subgroup is normal, the 2-element group F_2 / H is cyclic of order 2. By counting on the ways that $f: F = ?a?*?b? \to F_2 / H \cong \mathbb{Z}_2$ sends generators a and b to $[0]$ and $[1]$, one can classify these subgroups. The surjectivity of the canonical projection f implies that there are only three possible compositions: (1) $a \mapsto [0], b \mapsto [1]$, (2) $a \mapsto [1], b \mapsto [0]$, (3) $a \mapsto [1], b \mapsto [1]$. For (1) one sees a is send to identity, so a should be in the quotient; b is send to the element in $\mathbb{Z}_2$ whose square is identity, so b^2 should be in the quotient; and element aba^{-1} is added to make it the quotient indeed a normal subgroup, i.e., it is added to make H a normal closure of $\{a, b^2\}$. Thus, (1) corresponds to the subgroup,

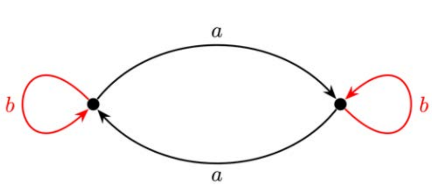
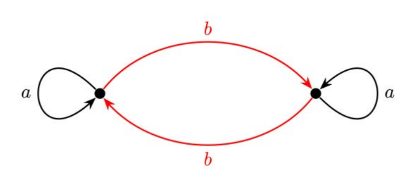
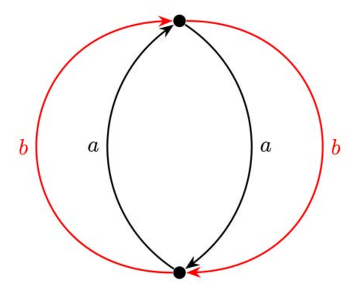
$$H_1 = ?b, a^2, aba^{-1}?. \quad (3)$$

Similarly, one finds,

$$H_2 = ?a, b^2, bab^{-1}?, H_3 = ?a^2, b^2, ab?. \quad (4)$$

Since each H_i is normal in F_2, each of them forms a single conjugacy class. Then use proposition 2 to find corresponded covering space X_{H_i} (the process of finding X_{H_i} in proposition 2 is constructive) (table 1):

Table 1. Connected 2-sheeted coverings of bouquet of 2 circles

H_1	H_2	H_3
		

Remark 1: Suppose one sets the vertex of $S^1 \vee S^1$ as x and two vertices $y_1, y_2 \in p^{-1}(x)$. Then $p_*\left(\pi_1\left(X_{H_1}, y_1\right)\right) = p_*\left(\pi_1\left(X_{H_1}, y_2\right)\right)$, as one can check that changing base point from left vertex y_1 of the above graph X_{H_1} to vertex y_2 on the right still get three generating loops b, a, aba^{-1}. This is simply a consequence of the symmetry of X_{H_1}. Therefore, the claim in conjugacy theorem (Lee, 2010. Theorem 11.34) is not violated [6]. Note that X_{H_2} and X_{H_3}, which are the first two graphs on the original table in Hatcher [3], are normal covers, which happens by proposition 3 if and only if the induced subgroups are normal in $\pi_1(X) = F_2$. And this is the same for X_{H_1}.

3.2 Connected 3-Sheeted Coverings of $S^1 \vee S^1$

This section finds all connected 3-sheeted covering spaces of $S^1 \vee S^1$, up to isomorphism of covering spaces without basepoints.

3.2.1 General combinatorial argument

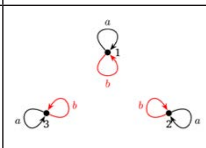
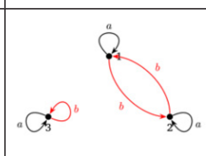
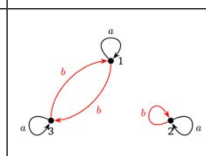
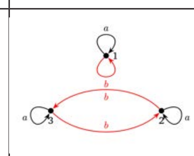
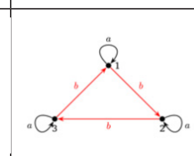
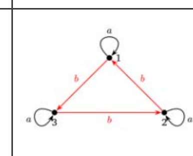
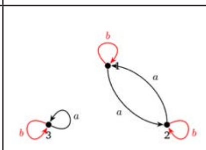
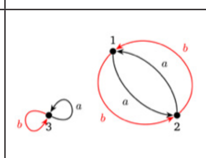
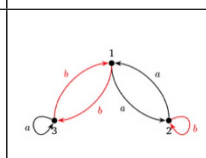
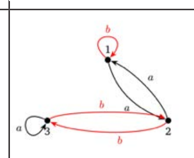
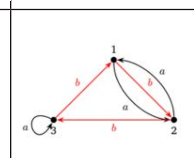
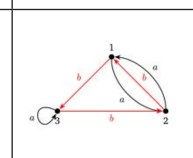
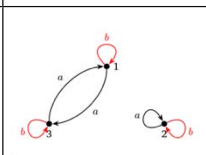
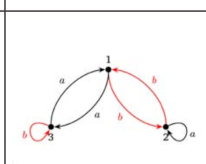
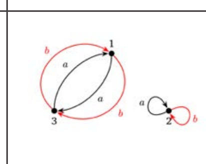
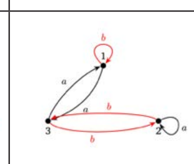
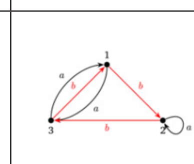
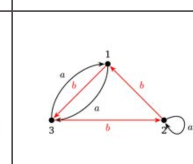
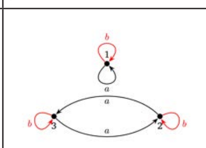
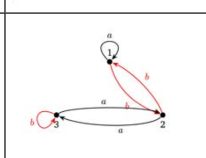
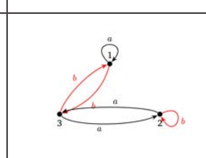
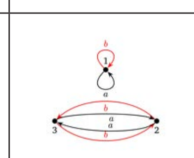
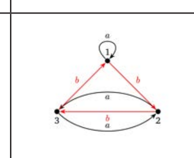
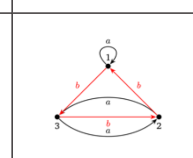
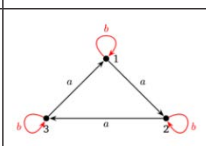
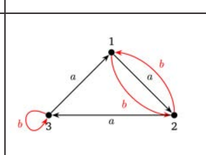
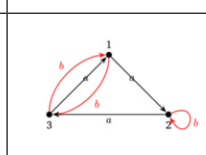
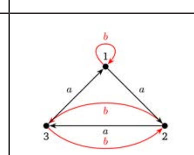
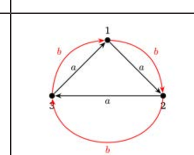
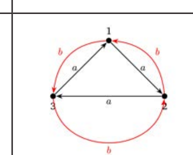
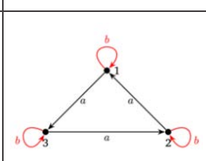
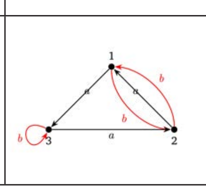
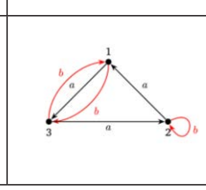
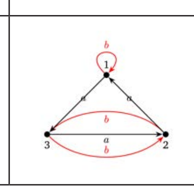
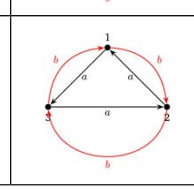
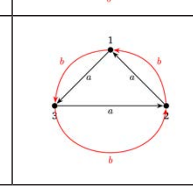
Apart from this algebraic approach, one may also use the approach by action on fiber, which is much simpler and will be used for the next classification. This is detailed explained in Hatcher: n-sheeted covering spaces of X are

ISSN 2959-6157

classified by equivalence classes of homomorphisms $\pi_1(X) \to S_n$. The theory uses the action of the fundamental group on the fiber of the covering map at basepoint, which consist of the ends of the liftings of the paths in the base space [3]. Therefore, one checks all possible ways of mapping the generators a and b of $\pi_1(X) = F_2 = ?a?*?b?$ to elements in symmetric group S_3 an classify them.

One can obtain Table 2 of all $6 \times 6 = 36$ possible combinations. Each row is a possible way of sending a to an element in $S_3 = \{(1),(12),(13),(23),(123),(132)\}$, and each column is a possible way of sending b to an element in S_3. This paper denotes the covering space, the 2-oriented graph, on i-th row and j-th column as $K(i, j)$.

Table 2. Connected 3-sheeted coverings of bouquet of 2 circles

a \ b	(1)	(12)	(13)	(23)	(123)	(132)
(1)						
(12)						
(13)						
(23)						
(123)						
(132)						

To proceed, one can use the same notation as in Hatcher: for vertex x_0 in the wedge of two circle, this paper uses $L_\gamma : p^{-1}(x_0) \to p^{-1}(x_0); \gamma(0) \mapsto \gamma(1)$ to denote one permutation on fiber $p^{-1}(x_0)$, where γ is the unique lift of the loop γ with given starting point $\gamma(0)$ [3]. Then $\gamma \mapsto L_\gamma$ gives a homomorphism φ from $\pi_1(X, x_0)$ to the group of permutation of $p^{-1}(x_0)$ which is the same as S_3. Then this paper exemplifies how the author obtains each graph $K(i, j)$ by example on $K(1,2)$.

The graph $K(1,2)$ is corresponded to the combination of $\varphi(a) = (1)$ and $\varphi(b) = (12)$. The loop a in X is sent to permutation $L_a = (1) = id_{p^{-1}(x_0)}$, the identity on $\{y_1, y_2, y_3\} = p^{-1}(x_0)$, i.e., each lift is a loop with same starting and ending point. This paper labeled vertices $y_{1,2,3}$ as $1,2,3$ on graph $K(1,2)$.

The black edges with orientations show the associated lifts of a. Similarly, the red edges with orientations show the associated lifts of b, as $L_b=(12)$ means the lift b starting with y_1 ends at y_2; the lift b starting with y_2 ends at y_1; and the lift b starting with y_3 ends at y_3.

In all, (1) means identity on the fiber, $(1,2)$, (13), (23) give transposition on the fiber, and (123), (132) give cyclic permutation. This paper then obtain the table above. One sees that there are 10 of them disconnected:

$$K(1,1),K(1,2),K(1,3),K(1,4),K(2,1),K(2,3),\\ K(3,1),K(3,3),K(4,1),K(4,4) \quad (5)$$

3.2.2 Analysis of connected subcollection

The following is the analysis of the remaining 26 covering spaces.

First see that $K(4,3)$ has the same induced subgroup as $K(3,4)$: by choosing the basepoint at y_3 this paper directly reads off all generating loops of $K(4,3)$ to obtain $?a^2,b^2,aba^{-1},bab^{-1}?$. With the same basepoint at y_3 one can directly read off all generating loops of $K(3,4)$ to obtain $?b^2,a^2,bab^{-1},aba^{-1}?$. Therefore, their induced subgroups are the same, denoted as H_{3443} and the two covers are put into the same basepoint-preserving isomorphism class of coverings.

$K(3,2)$ and $K(4,2)$ have the same induced subgroups $H_{342}=?b,a^2,ab^2a^{-1},abab^{-1}a^{-1}?$ by choosing basepoint y_3. $K(3,2)$ and $K(4,2)$ are classified as the same basepoint-preserving isomorphism class of coverings.

$K(2,3)$ and $K(2,4)$ have the same induced subgroups $H_{234}=?a,b^2,ba^2b^{-1},baba^{-1}b^{-1}?$ by choosing basepoint y_3. $K(2,3)$ and $K(2,4)$ are classified as the same basepoint-preserving isomorphism class of coverings.

Now observe that H_{3443},H_{342},H_{234} are all conjugate to each other because these covering spaces X only differ by basepoints y_1,y_2,y_3. Denote this conjugacy class by C_1.

By similar argument one can list the tuples of the following form: two graphs in the same isomorphism class, basepoint y_3,induced subgroup, conjugacy class. They are,

$$\left(K(5,1),K(6,1),y_3,H_{561}=?b,a^3,aba^{-1},a^{-1}ba?,C_2\right) \quad (6)$$

$$\left(K(5,2),K(6,2),y_3,H_{562}=?b,a^3,ab^2a^{-1}?,C_3\right) \quad (7)$$

$$\left(K(5,3),K(6,4),y_3,H_{563}=?b^2,a^3,aba^{-1}?,C_3\right) \quad (8)$$

$$\left(K(5,4),K(6,3),y_3,H_{564}=?b^2,a^3,a^{-1}ba?,C_3\right) \quad (9)$$

$$\left(K(1,5),K(1,6),y_3,H_{156}=?a,b^3,bab^{-1},b^{-1}ab?,C_4\right) \quad (10)$$

$$\left(K(2,5),K(2,6),y_3,H_{256}=?a,b^3,ba^2b^{-1}?,C_5\right) \quad (11)$$

$$\left(K(3,5),K(4,6),y_3,H_{356}=?a^2,b^3,bab^{-1}?,C_5\right) \quad (12)$$

$$\left(K(4,5),K(3,6),y_3,H_{456}=?a^2,b^3,b^{-1}ab?,C_5\right) \quad (13)$$

$$\left(K(5,5),K(6,6),y_3,H_{5656}=?a^3,b^3,ab^{-1},b^{-1}a?,C_6\right) \quad (14)$$

$$\left(K(5,6),K(6,5),y_3,H_{5665}=?a^3,b^3,ab,ba?,C_7\right) \quad (15)$$

There are in total 13 distinct subgroups of $\pi_1(X)$, corresponded to 13 isomorphism classes if insisting upon preservation of basepoint at y_3. They are,

$$H_{3443},H_{342},H_{234},H_{561},H_{562},H_{563},H_{564},H_{156},H_{256},\\ H_{356},H_{456},H_{5656},H_{5665} \quad (16)$$

To see there are no more than 13 subgroups, evoke the formula in Theorem 2. In particular $N(1,2)=1$, $N(2,2)=3$ and $N(3,2)=3(3!)^{2-1}-2\cdot N(1,2)-N(2,2)$ $=13$. If this paper relaxes on the basepoint preservation condition, one gets 7 isomorphism classes of connected covers with corresponded 7 conjugacy classes,

$$C_1,C_2,C_3,C_4,C_5,C_6,C_7 \quad (17)$$

Graphs of the representatives of these isomorphism classes are shown in Table 2.

3.3 X-digraphs of Finite-index Subgroups of F_2

This section briefly discusses some implications of the graphs and groups this paper finds in section 3 and 4 in context of combinatorial framework introduced by Kapovich and Alexei [4].

Definition 2: An X-digraph Γ with a finite set $X=\{x_1,\dots,x_N\}$ is a combinatorial graph $\hat{\Gamma}=(V,E,\mu)$ with $V(\Gamma)=V(\hat{\Gamma})$ and E is equipped with orientations, i.e., directed, and labels $\mu(e)\in X$ for each $e\in E$.

From this simple concept, the paper derives the classical subgroup graph $\Gamma(H)$ from a subgroup H of a group $F(X)$ freely generated by X. The definition evokes the uniqueness of such derivation with certain properties and is thus not included here.

ISSN 2959-6157

Definition 3: An X-digraph Γ is said to be X-regular if for every vertex v of Γ and every $x \in X \cup X^{-1}$ there is exactly one edge in $\hat{\Gamma}$ with starting vertex v and label x.

In the case $F_2 = F(X)$ with $X = \{a,b\}$, there are some immediate consequences. First, all of the subgroups in section 3 and 4 are finitely generated by their explicit expressions. Then by Lemma 5.4 in Kapovich and Alexei [4], their subgroup graphs are finite. Second, these subgroup graphs are finite X-regular graphs because of their finite index and Proposition 8.3 in Kapovich and Alexei [4].

It is easy to conjecture that the 2-sheeted and 3-sheeted coverings are perhaps related to the subgroup graphs of the corresponded finitely generated subgroups of F_2: the second part of the Proposition 8.3 in Kapovich and Alexei states that $|F(X):H| = \#V\Gamma(H)$ for any $H \leqslant F(X)$ and is verified by the fact that the 2-sheeted covering graphs have 2 vertices and 3-sheeted covering graphs have 3 vertices. However, this identification is obviously not true. For example, the 2-sheeted covering X_{H_3} for subgroup H_3 is 4-valent with both labels in $X = \{a,b\}$ doubly entering and exiting, violating the condition for folded-ness in definition of subgroup graph. More subtle relationships remain to be explored.

4. Universal Covering of $S^1 \vee S^1$

For completion, this section discusses the universal covering of $S^1 \vee S^1$ and some of its geometric-group-theoretic aspects. The general theory of associating a group $G = ?g_\alpha \mid r_\beta?$ with a 2-complexes X_G and X_G such that $\pi_1(X_G) \cong G$ and the orbit space $X_G / G \cong X_G$ with action by left translation can be seen for example in [3]. Since $G = F_2 = ?a,b?$ is freely generated by 2 generators a,b, there are no relators. Then X_G is just the Cayley graph $K = Cay(G,S)$ with $S = \{a,b\}$, and X_G is just the base space $S^1 \vee S^1$.

One also has a properly discontinuous action,

$$G \curvearrowright K; g \cdot x = gx \quad (18)$$

And thus a normal covering,

$$p: K \to K/G \approx S^1 \vee S^1 \quad (19)$$

Due to Hatcher proposition 1.40 (a) [3]. The action is also free due to proposition 4.1.10 in Löh [7]. $Cay(F_2)$ serves as the fundamental example of many geometric and topological theories on group. For example, as a tree, it has its geometric realization a hyperbolic space and thus a $CAT(0)$-space, and the underlying group is an example of Gromov's hyperbolic group [7, 8]. It is also well-known that free group of rank at least 2 has its ends a Cantor set, and the ends coincide with Gromov boundary for the Cayley graphs of free groups are tree [7]. Note that the ends and Gromov boundary are defined for finitely generated group by their Cayley graphs because change of generating set gives quasi-isometry of the space, which preserves the ends and Gromov boundary up to homeomorphism. In the context of hyperbolic groups, the Gromov boundary often supports a natural visual metric, which can give rise to measures like the Patterson-Sullivan measure (Figure 1). These measures help people understand the distribution of geodesics or group actions on the boundary in a quantitative way, similar to how fractal measures work in fractal geometry [9, 10].

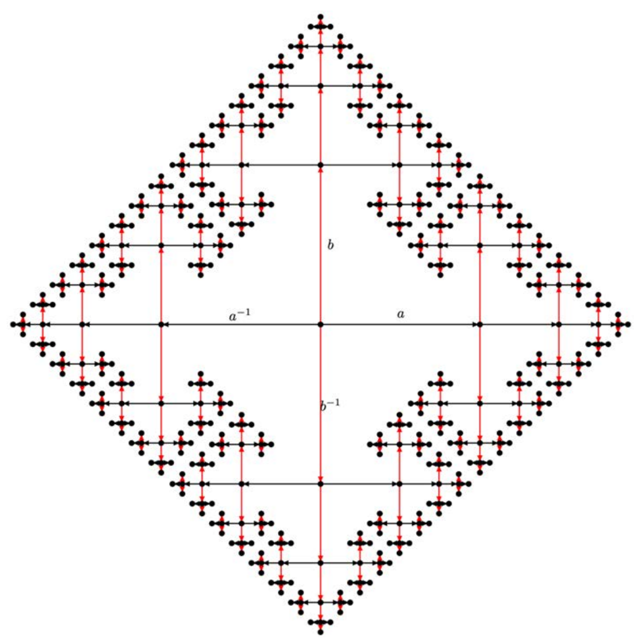


Fig. 1 Cayley graph of F_2, with the same coloring as in section 3, 4.

5. Conclusion

The paper classified the connected 2-sheeted and 3-sheeted covering spaces of the bouquet of two circles. The key computation relies on the classification theorem (Theorem 1), establishing a correspondence between covering spaces and subgroups of the rank-2 free group. The enumeration of these coverings provides insights into the universal covering. Additionally, the X-digraph serves as another powerful tool to represent these subgroup, making rich combinatorial computations accessible. It will be interesting to see how the covering graphs may relate to these classical combinatorial graphs. Such interplays between graphs and groups are fundamental theme in geometric group theory from which this paper finds broader contexts. For example, researches generalized Rips's theorem that torsion-free hyperbolic groups admit finite Eilenberg-MacLane spaces. It showcases the depth and complexity of the relationships between group actions on trees and the classification of spaces, reinforcing the significance of understanding covering spaces in topological settings.

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