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The Generalization of Cauchy Integral Formula

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Abstract:

The Cauchy Integral Formula is a fundamental result in complex analysis, providing a powerful method for evaluating analytic functions within a closed contour. This paper explores the Cauchy Integral Formula in its basic and higher-order forms. The basic Cauchy Integral Formula allows for the computation of a function's value at a point inside a closed contour by an integral over the contour. The higher-order form extends this to compute the derivatives of analytic functions. The Cauchy-Goursat Theorem will be discussed, which generalizes the Cauchy Integral Formula by addressing functions with singularities inside the contour. Additionally, the paper covers the generalized Cauchy Integral Formula involving contours that enclose infinity, providing insights into its application for evaluating functions at such singularities. Practical examples illustrate the theoretical concepts, demonstrating the application of these formulas in various scenarios. This study underscores the significance of these integral formulas in complex analysis and their utility in solving complex function problems.

Keywords: Cauchy integral formula; extension; integration.

1. Introduction

All Complex analysis is a rich and profound field of mathematics that explores functions of complex variables. Among its many powerful functions, the Cauchy Integral Formula stands out as a cornerstone result, offering a direct method for evaluating analytic functions. Established by Augustin-Louis Cauchy in the 19th century, this formula relates the value of an analytic function at a point to an integral over a closed contour surrounding that point [1]. Its elegance lies in its ability to express function values and derivatives purely in terms of contour integrals [2]. The Cauchy Integral Formula has been extended to higher-order derivatives, providing crucial insights into the behavior of analytic functions. This extension, often referred to as the higher-order Cauchy Integral Formula, is particularly useful in various applications where calculating derivatives directly may be challenging [3][4]. By expressing the derivatives in terms of contour integrals, this formula simplifies many complex analytical problems [5]. Additionally, the Cauchy-Goursat Theorem, an important generalization of the Cauchy Integral Formula, allows for the evaluation of integrals involving functions with

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singularities inside the contour. This theorem broadens the scope of the original formula, making it applicable to a wider range of problems in complex analysis [6]. The Cauchy-Goursat Theorem underscores the importance of residues and singularities in understanding complex functions. Furthermore, the generalized Cauchy Integral Formula addresses cases where the contour encloses an infinite point, offering a method to evaluate functions at such singularities. This extension is essential for understanding the behavior of functions at infinity and has implications in several areas of mathematical analysis [7].

In recent years, the research on the Cauchy integral formula has been continuously expanded in the field of complex analysis. In particular, in the analysis of multiple complex variables, the multidimensional generalization of the Cauchy integral formula has been deeply explored, providing a new solution to high-dimensional complex analysis problems [8]. Similarly, when complex functions are applied to communication engineering, it can provide a solid foundation for it [9]. How to use the Cauchy integral formula to solve the case of more than two singular points is also a frequently studied topic [10].

This paper aims to explore these integral formulas, illustrating their theoretical foundations and practical applications through specific examples. The versatility and significance of these tools in complex analysis are highlighted.

2. Cauchy Integral Formula and Higher Order Form

The Cauchy integral formula is an important tool in complex analysis. Its content describes how the value of an analytical function in a single closed curve is expressed by the integral. Before expressing it through the formula, let f(z) be an analytical function on the complex plane, C be a simple closed curve around point a, and f(z) be analytical in C. Then the Cauchy integral formula is

$$f(z)=\frac{1}{2\pi i}\oint\frac{f(z)}{z-a}dz \tag{1}$$

The Cauchy integral formula also has a higher-order form, which is usually used to calculate the higher-order derivatives of functions. Its specific formula is:

$$f_{(a)}^{(n)}=\frac{n!}{2\pi i}\oint_c\frac{f(z)}{(z-a)^{n+1}}dz \tag{2}$$

Higher-order functions play an important role in both complex analysis and advanced mathematics, especially when dealing with complex functions and their reciprocals [1].

Example 1. Calculate the value $f(z)=e^z$ of at a=0. Use the unit circle $|z|=1$ as the path. According to the Cauchy Integral Formula:

$$f(0)=\frac{1}{2\pi i}\oint_C\frac{e^z}{z}dz \tag{3}$$

Since e^z is analytic inside the unit circle, the result of the integral is $e^0=1[3]$.

Example 2. Compute $f^{(2)}(0)$ where $f(z)=\sin(z)$. Use the unit circle C. According to the higher-order Cauchy Integral Formula:

$$f^{(2)}(0)=\frac{2!}{2\pi i}\oint_c\frac{\sin(z)}{z^3}dz \tag{4}$$

Since the Taylor Series expansion of sin(z) is $\sin(z)=z-\frac{z^3}{6}+\dots$, it follows that $f^{(2)}(0)=-1$ [4].

3. Generalization of Cauchy's Integral Formula

Definition 1. (Cauchy-Goursat theorem)

The Cauchy-Goorsat theorem is an important generalization of the Cauchy integral formula. In the Cauchy integral formula, we require the function to be analytic within the closed curve. If the function has unanalytic points within the closed curve, the Cauchy integral formula will not work. In this case, the Cauchy- Goursat theorem can be used. The Cauchy-Goursat theorem describes that there is a function in the complex plane, and within the closed curve C, the integral of the function is

$$\oint_c f(z)dz=2\pi i\Sigma_k Res(f,a_k) \tag{5}$$

The point a_k is an isolated singular point in the range of C, and $Res(f(z),a_k)$ is the residue of the function at the point [3]. This theorem provides a method to obtain the integral of a function by solving the residue, which contributed to the subsequent theoretical research and application of analytic functions.

Example 1. Compute the integral of $f(z)=\frac{1}{(z-1)^2}$ around the unit circle enclosing z=1. According to the Cauchy-Goursat Theorem, the result of the integral is:

$$\oint_C\frac{1}{(z-1)^z}dz=2\pi i\cdot Res\left(\frac{1}{(z-1)^2},1\right) \tag{6}$$

Since the residue is 1, so the integral value evaluates to $2\pi i$ [6].

Example 2. Let $f(z)=\frac{e^z}{(z-2)}$. Calculate the integral around the unit circle that encloses z=2. According to the

Cauchy-Goursat theorem, the result of the integral is:

$$\oint_C \frac{e^z}{(z-2)}dz = 2\pi i \cdot Res\left(\frac{e^z}{(z-2)},2\right) \quad (7)$$

By evaluating, we know that the residue is e^2, and the integral evaluates to $2\pi ie^2$.

Cauchy integral formula with infinite point

There is an analytical function $f(z)$ on the complex plane. If there is a closed path γ, in this closed path, this function contains the point at infinity [4]. It can be expressed in a common form:

$$f(z) = \frac{1}{2\pi i}\oint_\gamma \frac{f(C)}{C-z}dC \quad (8)$$

In this formula , the path $\gamma$is a circular path containing the point at infinity. In order to evaluate the integral at infinity, we can consider letting the radius R of the path approach infinity [3]. Expressed in equations, we can write:

$$\lim_{R\to+\infty}\frac{1}{2\pi i}\oint_{|z|=R}\frac{f(C)}{C-z}dC = f(z) \quad (9)$$

Where $|z| = R$ represents a path with a radius of R.

Example 3. Let $f(z) = e^z$. Compute $f(0)$ using the generalized Cauchy Integral Formula with the point at infinity.

Solution:

Firstly, we need to choose the contour: use a circle of radius R centered at the origin, and take the limit $R \to +\infty$.

Secondly, we need to apply the formula:

$$f(0) = \lim_{R\to\infty}\frac{1}{2\pi i}\oint_{|z|=R}\frac{e^C}{C-0}dC \quad (10)$$

Additionally, we need to compute the integral:

$$\frac{1}{2\pi i}\oint_{|z|=R}\frac{e^C}{C}dC \quad (11)$$

By parameterization, let C= $Re^{i\theta}$, then $dC = iRe^{i\theta}d\theta$. Then the integral becomes:

$$\frac{1}{2\pi \mathrm{i}}\int_0^{2\pi} e^{\mathrm{Re}^{\mathrm{i}\theta}} iRe^{\mathrm{i}\theta}\mathrm{d}\theta \Big/ Re^{\mathrm{i}\theta} = \frac{1}{2\pi}\int_0^{2\pi} e^{Re^{i\theta}}\mathrm{d}\theta \quad (12)$$

Finally, we complete the limit processing: as $R \to +\infty$, $e^{Re^{i\theta}}$ grows exponentially, and the integral evaluates to $e^0 = 1$, thus:

$$f(0) = \frac{1}{2\pi}\int_0^{2\pi} 0^{Re^{i\theta}} d\theta = 1 \quad (13)$$

Therefore, $f(0) = e^0 = 1$.

Example 4.

Similarly, we can use the same process to evaluate another function $f(z) = \frac{1}{z^2+4}$. The Cauchy Integral Formula with the point at infinity will be used to compute $f(1)$.

Firstly, choose a circle $|z| = R$, and take the limit $R \to +\infty$.

Then, applying the formula leads to that

$$f(1) = \lim_{R\to+\infty}\frac{1}{2\pi i}\oint_{(z)=R}\frac{\frac{1}{C^2+4}}{C-1}dC \quad (14)$$

Substituting $f(z) = \frac{1}{z^2+4}$:

$$\oint_{|z|=R}\frac{1}{(C_+^2 4)(C-1)}dC \quad (15)$$

For a large enough R, the integrand is dominated by the behavior at infinity, where $\frac{1}{C_+^2 4}$ is effectively constant.

Finally, $f(1) = \frac{1}{1^2+4} = \frac{1}{5}$.

4. Conclusion

In this paper, it has examined the Cauchy Integral Formula and its higher-order forms, showcasing their pivotal role in complex analysis. The basic Cauchy Integral Formula allows for the evaluation of analytic functions at a point using a contour integral, while the higher-order version facilitates the computation of derivatives, offering a method to analyze more complex behaviors of functions. It also discussed the Cauchy-Goursat Theorem, which extends the application of the integral formula to cases involving singularities within the contour. This theorem broadens the scope of the Cauchy Integral Formula, making it applicable to a wider range of problems in complex analysis. The generalized Cauchy Integral Formula for contours enclosing infinity provides a valuable tool for evaluating functions at infinite points. Through practical examples, it demonstrated how these theoretical results can be applied to solve specific problems, highlighting their utility in real-world scenarios. Overall, the Cauchy Integral Formula and its extensions are fundamental tools in complex analysis, providing deep insights into the properties and behaviors of analytic functions. Their applications extend beyond theoretical mathematics to various practical problems, underscoring their importance in the field. Future research could explore further applications and generalizations of these integral formulas, continuing to advance our understanding of complex functions.

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References

[1] Ahlfors L. V. Complex Analysis. McGraw-Hill Education, (1979).

[2] Yi C., Pan H. Cauchy integral formula and its application in integral [J]. Journal of jiangxi normal university: natural science edition, 2010, (1): 4.

[3] Stein E. M., Shakarchi R. Complex Analysis. Princeton University Press, 2003.

[4] Lang S. Complex Analysis. Springer, 1993.

[5] Churchill R. V., Brown J. W. Complex Variables and Applications. McGraw-Hill Education, 2016.

[6] Bergweiler W., Eremenko A. Complex Analysis: An Introduction. Springer, 2003.

[7] Zhao T., Wei J., Chen Z., et al. Cauchy integral formula containing area at infinity and promote [J]. Journal of Yangtze university (since), 2015, 12(28): 1-4.

[8] Wang Z., Zhang W., He W. Extension and application of Cauchy integral Formula [J]. Journal of Xianyang Normal University,2018,33(06): 47-49.

[9] Ken P. The application of complex function in communication engineering [J]. Information recording materials, 2021, 22 (01): 51-52.

[10] Song Y. Calculation of perimeter integrals with more than two singularities [J]. Silk Road Vision, 2018, (5):1.