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The Exposition and Proof of Euler's Formula

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Abstract:

In the 18th century, Euler's formula (EF) was discovered by Leonhard Euler, and it has since been recognized as one of the most famous and beautiful equations in the mathematical world, occupying an extremely important place in numerous fields. The theory of complex functions has been significantly enriched by this formula, as it extends the domain of trigonometric functions to complex numbers and establishes a relationship between trigonometric functions and exponential functions. This paper explores the extended applications of Euler's formula in complex numbers, geometry, topology, and graph theory. The proof of Euler's formula in its algebraic form is achieved through the utilization of the Maclaurin series and the method of separation of variables, starting with the derivatives of functions. Additionally, the implications of the formula are investigated, with related proofs and applications being examined in plane geometry, graph theory, and physics. It is highlighted that Euler's formula not only occupies a significant position in mathematics but has also been a driving force behind the continuous development and innovation of modern technology.

Keywords: Euler's Formula; Exposition and Proof; Four-Color Problem.

1. Introduction

In 1748, the Euler formula was first mentioned by Leonhard Euler in a book, with his proof mainly based on the demonstration that the infinite series expansions on both sides of the equation were equal. However, it was not Euler who first proposed this formula, as in 1714, the British mathematician Roger Cotes had already provided the initial proof. Despite this, neither Euler nor Cotes had foreseen the geometric interpretation of the formula, which treats complex numbers as points on a complex plane—a concept that was not introduced until the work of the Danish-Norwegian mathematician Caspar Wessel in 1799 [1]. Wessel is widely recognized as the origin of the complex plane concept [1].

The Euler formula has found significant applications across various fields. For example, in the context of Laplace transforms, the introduction of Euler's formula simplifies the integration process [2]. In alternating current circuits, the geometric meaning of Euler's formula is utilized to demonstrate the intensity of electric current [3]. In communication engineering, Euler's formula enables the transformation of signals into complex numbers, thereby facilitating spectral analysis, filtering, and other operations, thus making signal processing and control possible [1]. The Euler formula has played a role in the prosperity of modern civilization.

Nevertheless, a comprehensive proof of the Euler formula is lacking, resulting in many people being familiar with its name but not its meaning. This thesis aims to provide a summary and discussion of the proof of Euler's formula, employing knowledge of Maclaurin series, differential derivatives, etc., and to research the generalization of the formula. The objective is to allow readers to gain a profound understanding of the mathematical principles underlying Euler's formula.

2 Basic Knowledge

The real number system is extended by complex numbers, allowing for the definition of negative numbers' square roots. A negative number is generally shown as follows: i is the imaginary unit, b is the imaginary part, and an is the real part. The distance between a point in the complex plane and the origin is represented by the modulus of a complex number. The angle formed by a complex number and the real axis's positive direction is known as its argument., usually denoted as θ.

The Taylor series is employed to represent an infinitely differentiable function at a point as an infinite series around that point. The general form of the Taylor series is $f(x) = f(x_0) + f(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots,$ and it becomes the Maclaurin series when a=0.

Euler's formula also heavily relies on trigonometric functions. The sine function ($\sin$) is the proportion between a right triangle's hypotenuse and opposing side lengths. The cosine function (cos) is the proportion between a right triangle's hypotenuse and neighboring side lengths. The tangent function (tan) is the sine function divided by the cosine function. Trigonometric identities $\sin^2 + cos^2 = 1$. Both the sine and cosine functions are periodic functions, exhibiting regular wave-like patterns.

A "graph" in graph theory is made up of vertices, sometimes referred to as nodes, and the edges that join them. A graph is said to be planar if it can be depicted on a plane so that no two edges cross, maybe except their endpoints. If a graph can be depicted on a plane with no edges crossing, it is called a planar graph. Each face in a planar network is bordered by a sequence of edges, comprising an infinite external face and a finite internal face. Faces are regions that are bounded by vertices and edges. Undirected, directed, weighted, simple, and multigraph graphs are all considered planar graphs.

An Eulerian path in a graph is defined as a path that visits each edge exactly once. An Eulerian circuit is characterized as a special type of Eulerian path that forms a closed loop and visits each edge exactly once. An Eulerian path in a graph is defined as a path that visits each edge exactly once. An Eulerian circuit is characterized as a special type of Eulerian path that forms a closed loop and visits each edge exactly once. For a graph to have an Eulerian circuit, it is necessary and sufficient that the degree of every vertex is even. A graph has an Eulerian path if and only if precisely two vertices have an odd degree, serving as the start and end points, while all other vertices have even degrees.

Two special types of graphs are complete graphs and bipartite graphs. A complete graph is characterized by every pair of distinct vertices being connected by an edge. A bipartite graph is defined as one where the vertices can be divided into two disjoint sets such that every edge connects a vertex from one set to a vertex from the other set.

3. Results and Their Justification

3.1 Justification of Euler's Formula in Complex Function Theory

Proof of Euler's Formula. First method:

From,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, -\infty < x < \infty \quad (1)$$

Replace each term x in the expansion with ix, ($i^2 = -1$)

$$e^{ix} = 1 + \frac{ix}{1!} + \frac{ix^2}{2!} + \frac{ix^3}{3!} + \ldots,$$

$$i^2 = -1, i^3 = i^2 * i = -i, i^4 = i^3 * i = 1, \ldots \quad (2)$$

$$e^{ix} = 1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \ldots$$

$$= cosx + i\sin x \quad (3)$$

That is, $e^{ix} = cosx + i\sin x$; From,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \quad (4)$$

Replace each term x in the expansion with -ix,

$$e^{-ix} = 1 + \frac{-ix}{1!} + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + \frac{(-ix)^5}{5!} \ldots, -\infty < x < \infty$$

$$= 1 - i\frac{x}{1!} - \frac{x^2}{2!} + i\frac{x^3}{3!} + \frac{x^4}{4!} - i\frac{x^5}{5!} - \ldots$$

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$$=1-\frac{x}{2!}+\frac{x^2}{4!}-\frac{x^3}{6!}+\ldots-i(\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots)$$

$$=cosx-i\sin x, \quad (5)$$

That is,

$$e^{-ix}=cosx-i\sin x \quad (6)$$

Second method: (Separation of Variables Integration) Let $z=\cos x+i\sin x$ be a complex number, differentiate both sides with respect to x, which yields $\dot{z}=i(\cos x+i\sin x)$. Then integrate both sides, $Inz=ix+C$. Taking $x=0$, $c=0$ can be obtained, resulting in $e^{ix}=cosx+i\sin x$.[4]

3.2 Justification of Euler‘s Formula in Geometry

Proof Method: Consider any convex polyhedron with the number of vertices V, the number of faces F, and the number of edges E, satisfying the following relationship:

$$V+F-E=x \quad (7)$$

It remains to prove that x is always constant and x=2.

First, appropriately transform the vertices of the convex polyhedron onto a smooth spherical surface, calling these points nodes. Then, connect these nodes. As a result, the entire smooth spherical surface is divided into many regions. Each region corresponds to the face of the polyhedron. At this point, the number of nodes, connecting lines, and regions on the spherical surface are the V, E, and F of the convex polyhedron.

Select any node A on the spherical surface. When removing node A, also remove all the connecting lines emanating from A. The number of nodes on the sphere decreases by one, and the number of connecting lines reduced is always one more than the number of regions reduced (if K lines are removed, then K-1 regions are reduced). Therefore, the current number of nodes on the sphere is V-1, the number of regions is F-(K-1), and the number of connecting lines is E-K. That is, (V-1) + F-(K-1) - (E-K) = x. Continuing in this way, the number of nodes, regions, and edges on the sphere always remains constant, that is, $nodes+regions-edges=x$.

After V-1 steps, there will be only one node left on the sphere. At this point, the number of nodes on the sphere = 1, that of regions = 1, and that of connecting lines = 0. Thus, x = 2.

Using the EF for polyhedra without “holes”, it can prove the EF for polyhedra with “holes”.

$$V+F-E=2(1-N) \quad (8)$$

Here, N is the number of “holes”.

Proof: Suppose a polyhedron has N “holes” and the number of edges on each side designated for each “hole” is K_1, K_2, K_N, etc. When these N “holes” are sealed off with N . K_1, K_2, K_N, etc. edge double-sided thin surfaces, the original polyhedron is transformed into a new polyhedron without any “holes”. Let the original polyhedron have V vertices, F faces, and E edges. Then, the new polyhedron will have $V+(K_1+K_2+......+K_N)$ vertices, and the number of faces will be $F+2N$, and that of edges will be $E+(K_1+K_2+......+K_N)$. In line with the Euler formula for polyhedra without “holes”, it can get

$$V+(K_1+K_2+......+K_N)+(F+2N)-(E+K_1+K_2+......+K_N)=2 \quad (9)$$

Obviously, it has $V+F-E=2(1-N)$。

4. Extensions and Citations of Euler’s Formula

4.1 Applications in Topology

4.1.1 Topology and modern technology

Data networks can benefit from the application of topology. Image recognition applications of topology concentrate on the analysis of the geometric shapes and topological structures of images. The understanding of image content can be improved by the identification of topological features such as connected regions and holes within images. These features are essential for the classification and recognition of images. The design of the physical and logical structures of computer networks can be assisted by topology. Network topology, which determines the connections between nodes, impacts the performance, scalability, and robustness of the network. Topology is also utilized in data compression, where it is capable of identifying contours and boundaries within image compression. Topology holds significant importance in the field of physics as well.

Topology research can uncover forms that differ from conventional solids, liquids, and gases under specific conditions. Discoveries include new materials such as topological insulators, topological superconductors, and Weyl semimetals. Theories of topology are also employed to elucidate the behavior of superconductors and superfluids. For instance, topology has aided scientists in understanding how superconductivity and superfluidity can arise in thin layers under certain conditions.

4.1.2 Polyhedra and their classification

In architectural compositions, the utilization of polyhedra's geometric shapes is employed to create unique visual effects and structural efficiency. Hexagonal planes are utilized in architectural designs because of their common presence in nature and their structural efficiency. Buildings with special spatial effects, such as the National Aquatics Center, also known as the Water Cube, are created using polyhedral structures. Visually striking architecture is achieved through the employment of the geometric shapes and structural characteristics of polyhedra.

4.1.3 Application of the Euler characteristic in 3D printing

The Euler characteristic, a basic concept in topology, is utilized to describe the global properties of a topological space. In the realm of 3D printing, it is applied to analyze and optimize the topological structure of objects being printed. The shape of a 3D model can be optimized to fulfill specific printing requirements by analyzing its Euler characteristic. Errors such as incomplete bonding between layers may occur during the 3D printing process. The impact of these errors on the model's topological structure can be assessed by calculating the Euler characteristic of the printed object.

4.1.4 Application of Euler's formula in navigation and geographic information systems

Euler's formula is employed in map making to analyze the geometric structure of maps, which assists in the design of more efficient map layouts and improves the readability of maps. It is also used to analyze the connectivity and accessibility of different areas on a map, which proves to be highly beneficial for traffic planning and network analysis. In navigation systems, the formula is applied to analyze the connectivity of paths and optimize route planning. In geospatial analysis, Euler's formula is used to evaluate geospatial patterns and processes, such as urban expansion and deforestation. By analyzing the impact of these changes on the geospatial structure, a better comprehension and prediction of these processes can be attained.

4.2 Applications in Networks and Graph Theory

4.2.1 Modern network design and optimization

Graph theory, the mathematical foundation of complex network research, is considered to have originated with the study of the Seven Bridges of Königsberg problem by the mathematician Leonhard Euler in 1736. It has since been recognized that numerous complex systems can be abstracted into networks and described from a topological perspective.

The significance of transportation for economic and social development is evident. The planning, design, and maintenance of transportation networks can be informed by studying their topological characteristics. Research on transportation networks predominantly concentrates on railways, aviation, and urban public transportation, and can be broadly categorized into two types: one focuses on the topological properties of transportation networks, while the other builds upon these properties to further explore the reliability or robustness of networks and structural optimization. Topological statistics of complex networks are interpreted differently across various networks.

4.2.2 Applications in social network analysis

The analysis of the patterns of relationships between nodes in social networks can be assisted by Euler's formula, which reveals user behavior, preferences, or patterns of relationships among nodes within the network. Meaningful association rules can be generated by identifying frequently occurring itemsets within the data, leading to a better understanding of the network structure and dynamics.

In the field of communications, fluctuations in the signal transmission process can be analyzed and addressed with the aid of EF. The conversion and calculation of trigonometric identities are facilitated by associating expressions of trigonometric functions such as sine, cosine, and tangent with complex numbers, thereby solving practical problems in communication.

4.2.3 Application of EF in computer graphics

In computer graphics, calculations involving rotation and transformation are facilitated by Euler's formula. Operations such as translation, rotation, and scaling of objects in three-dimensional space are made possible through the application of EF to matrix operations within the field. This has enabled the rapid and efficient development of computer graphics in areas like game development, animation production, and virtual reality.

4.3 The Practical Connection Between the Four-Color Problem and Euler's Formula

4.3.1 Application of the four-color problem in map drawing

The four-color problem has its origins in the practice of map making, where it is typically required to use different colors to distinguish between adjacent regions on a map. Terrain and rock representations were drawn based on the four-color principle in a study conducted by Alexander [5]. The method was shown to enhance the readability and realism of maps, maintaining essential terrain features such as contour lines and shadow effects. It was reported that 82% of users expressed a high level of satisfaction with

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the new map representation method, finding it to be more readable, realistic, and similar to aerial photographs compared to traditional topographic maps [5].

4.3.2 Application of the four-color theorem in scheduling and resource allocation

Efficient resource allocation and work arrangement to prevent conflicts and overlaps are the main focuses of the four-color theorem's application in scheduling and resource allocation. The management and optimization of resources are effectively achieved by applying the concept of the four-color theorem, ensuring that tasks or jobs do not conflict or overlap, thereby improving work efficiency and quality. For instance, the four-color principle can be applied to resolve conflicts between courses in the scheduling of school timetables. Courses can be considered as regions on a map, with conflicts between courses being analogous to those between adjacent regions. Time conflicts between courses can be avoided by suitably assigning colors to each course, leading to a more reasonable and scientific timetable for students.

5. Conclusion

The connection between trigonometric functions and complex exponential functions is established by the Euler formula. The formula is not only significant in mathematical analysis but also in the theory of complex variables. The derivation of the Euler formula in graph theory and complex variable functions is primarily proven in this thesis. Two methods are employed to demonstrate the Euler formula within complex variable functions, whereas one method is used to prove the formula in graph theory. The Euler formula is found to be closely related to quantum mechanics and electromagnetism in physics, aiding in a profound understanding of the essence and laws of nature. In the field of engineering, it is applied to enhance the efficiency and stability of system design in signal processing and control theory. In computer science, the Euler formula is also utilized in the simplification of problem-solving processes through algorithm design and optimization. The mastery of the Euler formula and its applications is crucial for enhancing the quality of higher mathematics learning. It assists students in understanding the profound connection between complex numbers and trigonometric functions and provides a powerful tool for solving more complex mathematical problems. The Euler formula, with its concise form and profound mathematical implications, exhibits the beauty of mathematics. It embodies the unity, simplicity, and peculiarity of mathematics and implicitly contains the philosophical principles of the universe. Through in-depth study of the Euler formula, a better understanding of the essence and value of mathematics can be achieved, thereby advancing the development of mathematics and science.

Authors Contribution

All the authors contributed equally and their names were listed in alphabetical order.

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