Dean&Francis
ISSN 2959-6157

Attempts and Inferences of the Four-Color Theorem

Xuanhan Zhang

Jiangxi Normal University Affiliated High School, 330000, Jiangxi, China

jiangcheng061027@gmail.com

Abstract:
The Four-Color Theorem is a classic problem in graph theory, stating that any planar map can be colored using no more than four colors so that no adjacent regions share the same color. Since 1976, when Appel and Haken used computer assistance to prove this theorem, it has been considered solved. However, due to its complexity and the difficulty of manually verifying the proof, some mathematicians still have doubts. This study proposes a new logical approach to provide an alternative proof for the Four-Color Theorem. Using a combination of theoretical derivations and graph theory tools, the paper first analyzes the basic structures of planar graphs, then use inductive reasoning to confirm coloring patterns in small graphs, gradually extending the method to more complex graphs. It developed a specific algorithm that simplifies vertices and edges in planar graphs, showing that any complex planar graph can be reduced to a few basic shapes, allowing it to be colored with fewer than four colors. Our results show that this reasoning model can confirm the Four-Color Theorem for a range of planar graphs without computer assistance. Though not covering all graph types, this model offers new insights for future research, especially in simplifying large graphs.

Keywords: planar graph coloring, adjacency, inductive reasoning, topological structure

1. Introduction

The Four-Color Theorem is a classic proposition in graph theory, notable for its wide-reaching impact on mathematics, computer science, and practical applications. The theorem's core assertion is that any planar map can be colored using no more than four colors, ensuring that adjacent regions have different colors. The proof of the Four-Color Theorem not only advanced the fields of graph theory and topology but also sparked significant discussions about the role and validity of computer-assisted proofs in mathematical research.

The research history of the Four-Color Theorem is both long and challenging. For over a century after it was first proposed, many mathematicians attempted to prove it, but none succeeded. It wasn't until 1976 that American mathematicians Kenneth Appel and Wolfgang Haken successfully proved the theorem using computer assistance. This marked the first time in mathematical history that a proof relied on computer technology, representing a major shift in research methods. However, despite the wide acceptance of computer-assisted proof, its complexity and the impossibility of fully manual verification have led to doubts among some mathematicians. These doubts have driven further attempts and improvements. The study of the Four-Color Theorem holds not only academic significance but also practical value. The theorem is widely applied in fields such as cartography, network design, and radio frequency allocation, providing a theoretical foundation for solving similar coloring problems. Additionally, research on the Four-Color Theorem has advanced computer-assisted proof techniques, opening new avenues for solving mathematical problems, particularly those that cannot be manually verified [1, 2].

Despite the breakthrough of computer-assisted proof, manual reasoning still holds significant research value. On one hand, manual proofs can deepen our understanding of the theorem's internal logical structure. On the other hand, they can offer new insights for solving other similar problems. The current challenge lies in developing a more intuitive and concise reasoning model that can explain the Four-Color Theorem without the need for computers. Addressing this challenge is a key focus of contemporary research in graph theory.

This study aims to introduce a new reasoning method and attempt a manual proof of the Four-Color Theorem. The research methods include inductive reasoning, topological analysis, and the use of graph theory tools. By constructing a logical deduction model, this paper starts with simple planar graphs and gradually extends the approach to more complex cases, exploring whether it is possible to color planar graphs using four colors under certain constraints. The study includes a review of existing proofs, the construction of a reasoning model, and its application to specific graphs. Through this research, it hopes to offer new insights into manual proofs of the Four-Color Theorem, deepen our understanding of this classic problem, and provide inspiration for solving other complex graph theory issues.

2. Historical Background and Progress of the Four-Color Theorem

The Four-Color Theorem, a classic problem in graph theory, has been a topic of great interest in the mathematics community since it was first proposed by British mathematician Francis Guthrie in 1852. Guthrie's question was simple: can any planar map be colored using no more than four colors, ensuring that adjacent regions have different colors? Despite its straightforward formulation, proving the Four-Color Theorem turned out to be extremely complex and became a longstanding challenge for mathematicians [3].

One of the earliest attempts to prove the theorem came in 1879, when British mathematician Arthur Kempe proposed a method to solve it. However, other mathematicians soon discovered flaws in Kempe's proof, invalidating its correctness. Nonetheless, Kempe's work inspired further research, and many mathematicians began working to solve the problem, proposing several related theories. For example, the Five-Color Theorem is one significant result, which states that any planar map can be colored using up to five colors. Although it did not completely solve the Four-Color Theorem, it laid the groundwork for further study.

Over the next century, mathematicians used various mathematical tools, such as topology and combinatorics, to get closer to solving the problem. The challenge of the Four-Color Theorem lies in the need to verify all possible planar graphs, an enormous number that makes it impossible for traditional manual reasoning methods to exhaust. As mathematicians deepened their understanding of the problem, they realized that a manual proof of the Four-Color Theorem might be nearly impossible, especially when dealing with highly complex graphs.

It wasn't until 1976 that mathematicians Kenneth Appel and Wolfgang Haken, using computer-assisted technology, successfully proved the Four-Color Theorem. Their proof relied on large-scale computations, analyzing and verifying thousands of special cases to exhaust all possibilities. This was the first time in mathematical history that proof depended on computer technology, marking a groundbreaking moment and signaling a new era in mathematics where computers became essential tools for proving theorems.

Appel and Haken's computer-assisted proof generated significant controversy at the time. Traditionally, mathematical proofs were seen as purely logical reasoning processes that humans could directly understand and verify. However, computer-assisted proofs relied on complex programs and vast amounts of data, which made it difficult for mathematicians to manually verify the entire process, leading some to question the validity of the proof [4]. These concerns focused on two main issues: first, whether a computer could be considered a reliable tool for proofs, especially given the potential for programming errors or hardware issues; and second, whether mathematicians could fully trust the computer's calculations, particularly in cases where manual verification was impossible.

Despite these concerns, Appel and Haken's proof was widely accepted and became the official proof of the

ISSN 2959-6157

Four-Color Theorem. This computer-assisted proof opened a new chapter in the history of mathematics, prompting a reevaluation of the nature of mathematical proofs and driving the increasing use of computers in mathematical research. As technology continues to advance, computer-assisted proofs have become more common in other fields, especially when addressing problems that are too complex to be solved using traditional methods. In these cases, computer proofs have become indispensable tools.
However, despite the success of the computer-assisted proof in establishing the Four-Color Theorem, mathematicians still seek a more intuitive and simpler proof. A manual proof could deepen the understanding of the theorem's internal logic and avoid over-reliance on technological tools, which may present potential issues.
Currently, research on the Four-Color Theorem not only focuses on finding a simpler proof but also explores its applications in other fields. For example, graph coloring problems appear not only in map drawing but also in areas such as information science, network optimization, and frequency allocation [5]. The theoretical foundation of the Four-Color Theorem provides a powerful tool for solving these practical problems and has driven further development in the field of graph theory. Looking forward, research continues to explore how a simple and clear manual proof could be developed without the use of computers, which remains a challenge for mathematicians. Meanwhile, the extended applications of the Four-Color Theorem in other areas of graph coloring problems are also being explored. Through continued study of this classic problem, not only can mathematical theory be advanced, but deep impacts can also be made on fields such as information technology and network science.

3. Attempting a Proof of the Four-Color Theorem Based on Reasoning

After the breakthrough of computer-assisted proof, some scholars in the mathematical community continued to seek a purely manual reasoning proof for the Four-Color Theorem. This is because a manual proof not only deepens the understanding of the theorem's internal structure but also provides new solutions for similar problems. Current research focuses primarily on exploring the topological properties of planar graphs and using logical deduction to find a direct proof of the Four-Color Theorem. This section introduces an attempt to prove the Four-Color Theorem based on inductive reasoning. The core idea of this reasoning model is to analyze the coloring patterns of small planar graphs and gradually extend them to more complex graphs. The research method involves structured processing of planar graphs, specifically simplifying the vertices, edges, and faces of the graph to ensure that adjacency constraints during the coloring process are met.
It is known that all maps can generate a network, but not all networks formed by maps are valid. For example, if a map contains intersecting lines, as shown in the diagram when attempting to draw the network as a map, the drawing cannot be completed successfully unless the intersecting lines are "untangled," meaning the intersections must be resolved while maintaining correct connections to form a valid map.
Suppose a map consists of countries. According to Euler's theorem, there must exist a country with no more than five neighboring countries (such as a coastal country located at the edge of the map). Removing this country reduces the number of countries in the map to a-1, and at this point, the map can be colored using four colors. If this process is successful, it proves the Four-Color Theorem. Using this method, it is possible to prove that at least five colors are sufficient for coloring, but the Four-Color Theorem cannot be solved using the same method.
First, consider a simple case where a country is surrounded by four neighboring countries, as shown in Figure 1. This case is relatively easy to resolve. By drawing two intersecting paths, and since the intersections cannot coexist, two different solutions are obtained, both of which successfully use four colors for the coloring, as shown in Figure 1.

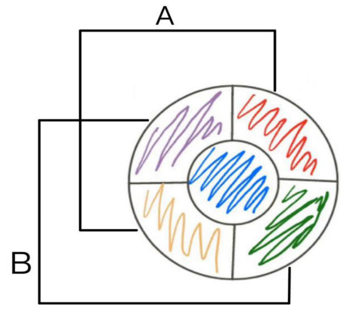

Fig. 1 Coloring the problem of the four neighboring countries with four colors (Photo/Picture credit: Original).

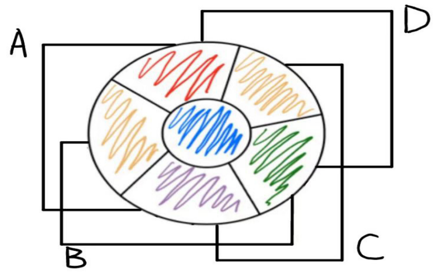

Fig. 2 Coloring the problem of the four neighboring countries with four colors (Photo/Picture credit: Original).

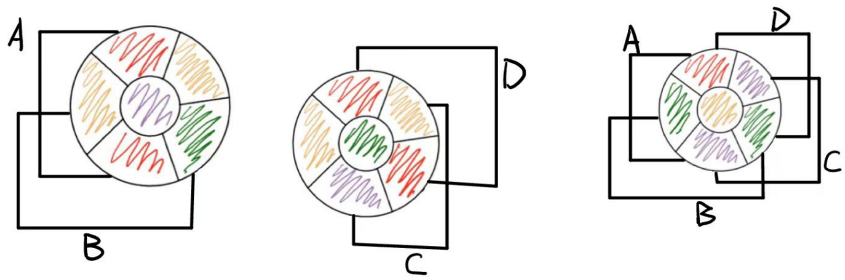

Fig. 3 Three ways to color the problem of the five neighboring countries with four colors (Photo/ Picture credit: Original).

The situation gets complicated when trying to solve a problem involving five neighboring countries. As shown in Figure 2, draw four channels, which combine the three situations shown in Figure 3:

1) If channel B is smooth, channel A is bound to be blocked. Both sides of A can be marked red and the middle area purple;

2) Similarly, channel C is unblocked and channel D is blocked;

3) If both channel B and channel C are blocked, channel A and channel D are unblocked. At this time, both sides of channel B are green, both sides of channel C are purple, and then the blue area is yellow. Complex substitution operations can still be solved using this method. However, this is only one case. After a follow-up review of the data, try to draw a new channel, the difference is that the AD channel crossing. Although the gap is small, it exposes the limitations of the approach. In this case, the problem cannot be solved by a color change, as shown in Figure 4.

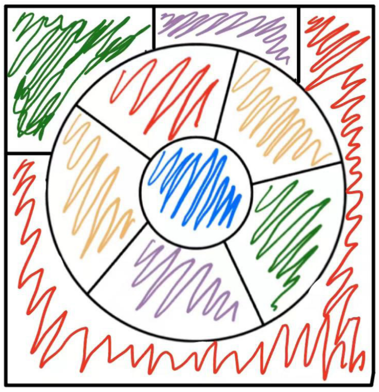

Fig. 4 A legend of a special case (Photo/ Picture credit: Original).

This was the dilemma Kemp faced in his study. With the increase of map complexity, the number of color changes is increasing, and simple repetition is difficult to solve the problem. This is how the computer finally succeeded in proving the four-color theorem. Tens of billions of repetitions proved the feasibility of the dyeing method.

The four-color theorem not only reveals the complexity of mathematical problems, but also demonstrates the importance of innovative approaches to solving mathematical problems. The unprecedented computer-aided technology was used in the proof process, which caused great controversy in the mathematical circle. Although some mathematicians questioned whether computer-aided proofs were true mathematical proofs, over time the vast majority of mathematicians came to accept the new method.

4. Conclusion

The conclusion drawn from this study is that, although the computer-assisted proof has fully resolved the Four-Color Theorem, manual proofs remain a worthwhile avenue of exploration. Inductive reasoning provides a new perspective for coloring simple graphs, demonstrating some feasibility, especially in the coloring of local regions. However, the limitations of this reasoning model become apparent when dealing with more complex graphs, indicating a need for further refinement. Our research results do not contradict the previous computer-assisted proofs, but it has identified the potential feasibility of manual reasoning, particularly when applied to smaller-scale graphs. Unlike previous studies that relied on computer verification, it explored a more intuitive method of reasoning, although it is still not comprehensive. This study offers a new angle for proving the Four-Color Theorem manually, but its contribution to the overall theory is limited, primarily paving the way for future research.

There are also some limitations to this study. First, the reasoning model struggles to provide a complete solution when applied to complex graphs, and its range of applicability is somewhat limited. Second, the focus of this research has been on theoretical analysis, with a lack of validation and expansion in practical application scenarios. In practice, research on the Four-Color Theorem holds direct relevance for areas such as map drawing and network design. Overall, this study offers an initial exploration

ISSN 2959-6157

into a manual proof of the Four-Color Theorem, laying the groundwork for future research and practical application. By exploring simpler and more intuitive methods of proof, it hopes that this paper will provide new insights into research on the Four-Color Theorem and its related fields. Lastly, it hopes this research will contribute to the development of studies on the Four-Color Theorem, further promote the exploration of manual reasoning methods, and inspire more meaningful discussions, particularly regarding the extended applications of this theorem.

References

[1] Appel K, Haken W. Every planar map is four colorable. Part I: Discharging. Illinois Journal of Mathematics, 1977, 21(3), 429-490.
[2] Appel K, Haken W. Every planar map is four colorable. Part II: Reducibility. Illinois Journal of Mathematics, 1977, 21(3), 491-567.
[3] Wilson R J. Four Colors Suffice: How the Map Problem Was Solved. Princeton University Press. 2002.
[4] Tymoczko T. The Four-Color Problem and Its Philosophical Significance. The Journal of Philosophy, 1979, 76(2), 57-83.
[5] Rogers R M. The Four Color Problem: The Journey to a Proof and the Results of the Study. 2020.