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Fourier Series and Its Application to Series Sums

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Abstract:

Fourier series has always been a very key part of infinite series problem, not only can solve many difficult partial differential equations problems, but also has extremely important applications in signal processing, thermodynamic statistical physics, quantum physics and other disciplines. Based on the basic mathematical analysis knowledge such as integral, this paper focuses on the theoretical basis of Fourier series and its application and extension in solving infinite series sums. The main part of the paper will focus on the derivation of mathematical formulas, accompanied by text explanations and the author's thinking. In this paper, the author proves the convergence theorem and the discriminant method of practical significance, and applies the theoretical knowledge to the concrete calculation of series. At the same time, the author has also found many problems, and from different angles to explore the possible ways to solve the problem of series. These basic calculation ideas and results are instructive to the application of Fourier series in other fields.

Keywords: Fourier series; Convergence theorem; Parseval equation.

1. Introduction

In mathematical analysis, the expansion and calculation of series is always an extremely important proposition. Fourier series is particularly classical and important here. The discussion of such series is not only of mathematical interest, but also has a strong physical background. It is an indispensable mathematical tool in engineering technology, especially in radio communication and digital processing. In view of the periodic phenomenon in many scientific and technological problems, scientists usually adopt the idea of Fourier series, which can decompose a relatively complex periodic wave into a series of simple harmonics with different frequencies, so as to solve the problem.

In the initial stages of the Fourier series, beginners can clearly see that the Fourier series has important applications in ordinary differential equations and wave equations, especially in the solution of boundary value problems and initial value problems. Many complex PDEs are difficult to solve directly, but by using the Fourier series and decomposing the complex functions in the equation into sums of sine and cosine, the solution process can be simplified [1]. With the development of technology, its applications in physics are gradually emerging. One important

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application of the Fourier series and is signal reconstruction. Any periodic signal (such as electrical signals, audio signals) can be decomposed into a sum of sine and cosine waves by the Fourier series. This sum not only helps people understand the composition of the signal, but also allows people to approximate the reconstruction of the signal by taking the first few terms of the sum [2]. In addition, application of Fourier series in quantum mechanics involves the representation of wave functions and the transformation of momentum space. In quantum mechanics, behavior of particles is often described by wave functions, which can be expanded in terms of sine waves of different frequencies by the Fourier series. Fourier expansion of the wave function can also be transformed into a representation in momentum space, thus providing convenience for analyzing the distribution of particle momentum [3]. In addition to these, the Fourier transform has many other applications [4,5].

The subsequent part of the paper will mainly introduce the theoretical basis of the Fourier series and several calculation methods for the sums of series. The author will, based on the basic form of the Fourier transform, explore the convergence issue of the Fourier series through ingenious calculation methods and present the fundamental methods for determining its convergence. Simultaneously, the author will also attempt to prove a representative equation in the Fourier theory - the Parseval's equation, as well as its generalization and application in diverse circumstances. Subsequently, the author will enumerate multiple distinct and representative examples of solving the sums of series, solve these problems by applying the basic theoretical methods just introduced, and concurrently compare them with other possible approaches.

2. Methods and theory

2.1 Convergence of Fourier series

For a function f, the Fourier expansion is [6]

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n cosnx+b_n\sin nx) \tag{1}$$

with

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)cosnxdx b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx \tag{2}$$

To be sure, this series is determined by function f, but to make sense of it, the paper need to discuss its convergence. The author shall introduce this series

$$S(x_0)=\frac{a_0}{2}+\sum_{k=1}^{n}(a_k coskx+b_k\sin kx). \tag{3}$$

By bringing in the expression of a_k and b_k shown in Eq. (2), Eq. (3) can be further simplified as

$$S(x_0)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\left(\frac{1}{2}+\sum_{k=1}^{n}cosk(x-x_0)\right)dx \tag{4}$$

By using trigonometric identities $1/2+\sum_{k=1}^{n}coskx=\sin\left(n+\frac{1}{2}\right)x/2\sin\frac{x}{2}$, It is concluded that

$$S(x_0)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\left(\frac{\sin\left(n+\frac{1}{2}\right)(x-x_0)}{2\sin\frac{1}{2}(x-x_0)}\right)dx \tag{5}$$

Taking periodicity into account, the integral expression is obtained as

$$S(x_0)=\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x_0+t)+f(x_0-t))\frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}dx \tag{6}$$

This important integral is called the Dirichlet integral and is the origin for the discussion of Fourier series convergence, function $\sin\left(n+\frac{1}{2}\right)t/2\sin\frac{t}{2}$ is called the Dirichlet core [7]. At the same time, with the help of Riemann-Lebesgue lemma (f can be integrated and absolutely integrated), $\lim_{n\to+\infty}\int_a^b f(x)cosnxdx=0$ and $\lim_{n\to+\infty}\int_a^b f(x)\sin nxdx=0$, the paper can get the localization theorem of Fourier series: Whether the Fourier series of f converges at point x_0 depends only on the behavior of f near point x_0. On this basis, various methods of judging point convergence are also established.

For example, a classical method of identification called Dini theorem: $f\in R[-\pi,\pi]$ For certain real number s, let

$$\phi(t)=f(x_0+t)+f(x_0-t)-2s \tag{7}$$

If there is $\delta>0$ such that the function $\phi(t)?t$ is integrable and absolutely integrable on $[0,\delta]$, then the Fourier series of f converges to s at x_0.

Based on Dini theorem, the paper would have other reliable theorems: If the function f of period 2π is piecewise differential on $[-\pi,\pi]$, then the Fourier series of f converges to $(f(x_0+0)+f(x_0-0))/2$ at every point x_0, and in particular at the continuous points of f, it converges to $f(x_0)$. Therefore, as long as f has a first derivative on $[-\pi,\pi]$, it can be expanded into a Fourier series, and from this point of view, the Fourier series is much better than the power series. The point convergence of Fourier series is a very difficult problem, the author can only use the existing limited methods to solve some series sum problems.

2.2 Parseval Equation

To deal with more complex problems in series sums, the author also needs to introduce Parseval equation. This equation comes from the square mean convergence of the Fourier series. For a given f and a positive integer n, the paper would like to figure out that what ϕ polynomial $T_n = \sum_{k=0}^{n} \alpha_k \phi_k(x)$ makes the norm

$$\| f - T_n \| = \sqrt{\int_a^b \left(f(x) - T_n(x) \right)^2 dx} \tag{8}$$

takes the minimum, which also means the squared mean error is minimum. After expanding, Eq. (8) can be further simplified as

$$\| f - T_n \| = \| f \|^2 - \sum_{k=0}^{n} c_k^2 + \sum_{k=0}^{n} \left(c_k - \alpha_k \right)^2 \tag{9}$$

It follows that if and only if $\alpha_k = c_k \left(k = 0, 1, \cdots, n \right)$, $\| f - T_n \|^2$ takes the minimum value: $\| f - \sum_{k=0}^{n} c_k \phi_k \|^2 = \| f \|^2 - \sum_{k=0}^{n} c_k^2$. And that gives Bessel inequality $\sum_{k=0}^{n} c_k^2 \leqslant \| f \|^2$, When the inequality is equal, it is just the Parseval equation

$$\sum_{k=0}^{n} c_k^2 = \| f \|^2 \tag{10}$$

In the traditional Fourier transform, Parseval equation is that

$$\frac{a_0^2}{2} + \sum_{k=1}^{n} \left(a_k^2 + b_k^2 \right) \leqslant \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) dx \tag{11}$$

After having Parseval equation, the relationship between the Fourier series and the original function is much clearer: $f \in R^2[-\pi, \pi]$, a_n and b_n are Fourier series of f with respect to the system of trigonometric functions, so f can be approximated by the partial and square average of its Fourier series, i.e., the Parseval equation.

Furthermore, the author shall write the Parseval equations for $f+g$ and $f-g$ respectively,

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \left(f(x) + g(x) \right)^2 dx = \frac{\left(a_0 + \alpha_0 \right)^2}{2} + \sum_{n=1}^{\infty} \left(\left(a_n + \alpha_n \right)^2 + \left(b_n + \beta_n \right)^2 \right) \tag{12}$$

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \left(f(x) - g(x) \right)^2 dx = \frac{\left(a_0 + \alpha_0 \right)^2}{2} + \sum_{n=1}^{\infty} \left(\left(a_n - \alpha_n \right)^2 + \left(b_n - \beta_n \right)^2 \right) \tag{13}$$

By subtracting the two formulas Eq. (12) and Eq. (13), the Parseval equation that generalizes to two different functions could be given

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) g(x) dx = \frac{a_0 \alpha_0}{2} + \sum_{n=1}^{\infty} \left(a_n \alpha_n + b_n \beta_n \right) \tag{14}$$

Using the conclusion Eq. (14) just now, the term-by-term integral theorem of Fourier series can be proved: $f \in R^2[-\pi, \pi]$, Its Fourier series is $f(x) \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n cosnx + b_n \sin nx \right)$. For any interval $[a, b]$ contained in $[-\pi, \pi]$,

$$\int_a^b f(x) dx = \int_a^b \frac{a_0}{2} dx + \sum_{n=1}^{\infty} \int_a^b \left(\alpha_n cosnx + \beta_n \sin nx \right) dx \tag{15}$$

It is worth noting that this theorem states that the Fourier series of f can be integrated terms by terms regardless of whether it converges or not, which is a special property of Fourier series.

3. Application

Fourier series expansions and their associated Parseval equations have important applications in finding series sums. Here are some typical examples.

3.1 Calculate $\sum_{n=1}^{\infty} \frac{1}{n^p}$ with p be Even

First, the Fourier series of x on $[-\pi, \pi]$ is calculated by the basic formula of Fourier expansion

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n} \sin nx \tag{16}$$

Then, according to $\left(x^2 \right)' = 2x$ and $f'(x) \sum_{k=1}^{\infty} \left(k b_k coskx - k a_k \sin kx \right)$ and combined with the convergence theorem mentioned in section 2, an expansion of x^2 can be obtained.

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} cosnx, x \in [-\pi, \pi] \tag{17}$$

Put $x = \pi$ in Eq. (17) and then get that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Then use $a_0 = \frac{\pi^2}{3}, a_n = 4(-1)^n \frac{cosnx}{n^2}, b_n = 0, n = 1, 2, \cdots,$ and

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Parseval equation $\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{1}{2}\left(\frac{2\pi^2}{3}\right)^2 + \sum_{n=1}^{\infty}\frac{16}{n^4}$. Thus the problem is solved: $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^2}{90}$.

As the same, for function $f(x) = x^3, x \in (-\pi,\pi)$, its Fourier expansion is

$$x^3 = 2\sum_{n=1}^{\infty}(-1)^n\left(6-\pi^2 n^2\right)\frac{\sin nx}{n^2}, x \in [-\pi,\pi]. \quad (18)$$

Using the Parseval equation $\frac{1}{\pi}\int_{-\pi}^{\pi} x^6 dx = \sum_{n=1}^{\infty}\left[2(-1)^n\left(6-\pi^2 n^2\right)\frac{1}{n^3}\right]^2$ and its simplification $\frac{2}{7}\pi^6 = \sum_{n=1}^{\infty}\left(\frac{4\pi^4}{n^2} - \frac{48\pi^2}{n^4} + \frac{144}{n^6}\right)$ and combined with $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^4}$, the answer is clear: $\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}$.

In fact, the mathematical definition of the Riemann ζ function was originally a sum of p-series, which is defined as follows: a complex number s, the actual part >1,

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}. \quad (19)$$

It can also be expressed as $\zeta(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{s-1}}{e^x - 1}dx$ .Over the region $\{s : Re(s) > 1\}$, this infinite series converges and becomes a holomorphic function. Euler considered the case of s being a positive integer in 1740, and Chebyshev extended this to $s > 1$. Bornhard Riemann realized that the ζ function can be extended by analytical extension to a holomorphic function $\zeta(s)$ defined over the complex field $(s, s \neq 1)$. This is also the function studied by the Riemann conjecture. A more general answer can be obtained by further applying the residue method: $\sum_{n\neq 0}\frac{1}{n^p}(p = 2,4,6,\cdots) = -\frac{2\pi i^p}{p!}\frac{d^p}{dz^p}\left(\frac{z}{e^{2\pi z}-1}\right)|z = 0$ [8]. At the same time, the paper can also discuss more about the generalization of P-series $\sum_{n=1}^{\infty}\frac{1}{\left(n^{2k} \pm a^{2k}\right)^m}$ [9].

3.2 Caculate $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}$

In this case, direct computation is obviously not feasible, and the author need to consider the Fourier expansion of $f(x) = sgnx = \begin{cases} 1x > 0 \\ 0x < 0 \end{cases}$. Its Fourier expansion is $a_n = 0$ and $b_n = \frac{2}{\pi}\int_0^{\pi}\sin nx = \frac{2}{\pi}\frac{1-(-1)^n}{n}$. Therefore, it is calculated that

$$f(x) = sgnx\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1} \quad (20)$$

By convergence theorem, when $0 < x < \pi$ $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1} = sgnx = 1$. And by putting $x = \pi/2$ in, the equation is proved: $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$. It can be seen from this problem that mastering the Fourier expansion of basic functions is crucial to the calculation of series sums.

3.3 Other Applications

3.3.1 Series 1

For $x \in (0,2\pi)$ and $a \neq 0$, prove that

$$e^{ax} = \frac{e^{ax}-1}{\pi}\left(\frac{1}{2a} + \sum_{k=1}^{\infty}\frac{acoskx - k\sin kx}{k^2 + a^2}\right) \quad (21)$$

It is easy to think of expanding e^{ax} over $(0,2\pi)$ as a Fourier series

$$a_n = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\cos nxdx = \frac{1}{\pi}\frac{e^{ax}(a\cos nx + n\sin nx)}{n^2 + a^2}\Bigg|_0^{2\pi} = \frac{a\left(e^{2a\pi}-1\right)}{\pi\left(n^2 + a^2\right)} \quad (22)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} e^{ax}\sin nxdx = \frac{1}{\pi}\frac{e^{ax}(a\cos nx - n\sin nx)}{n^2 + a^2}\Bigg|_0^{2\pi} = -\frac{a\left(e^{2a\pi}-1\right)}{\pi\left(n^2 + a^2\right)} \quad (23)$$

The answer can be obtained by substituting the above Eq. (22) and Eq. (23) into the formula of Fourier expansion. Reader needs to pay special attention to the integration techniques used in the calculation, and at the same time, using this equation, the value of the series $\sum_{n=1}^{\infty}\frac{1}{n^2 + a^2}$ can be found.

3.3.2 Series 2

For $0 < a < \pi$ and define function $f(x) = \begin{cases} 1, |x| < a, \\ 0, a \leqslant |x| < \pi \end{cases}$, the question is to find the sum of the series: $\sum_{n=1}^{\infty}\frac{\sin^2 na}{n^2}$

and $\sum_{n=1}^{\infty}\frac{cos^2 na}{n^2}$.

It is clear that

$$f\left(x\right)\frac{a}{\pi}+\sum_{n=1}^{\infty}\frac{2\sin na}{n\pi}cosnx \tag{24}$$

With $a_0=\frac{2a}{\pi},a_n=\frac{2\sin na}{n\pi},b_n=0$. According to Parseval equation Eq. (11) and Question 3.1, the answer is that

$$\sum_{n=1}^{\infty}\frac{\sin^2 na}{n^2}=\frac{a\pi-a^2}{2} \tag{25}$$

$$\sum_{n=1}^{\infty}\frac{cos^2 na}{n^2}=\sum_{n=1}^{\infty}\frac{1}{n^2}-\sum_{n=1}^{\infty}\frac{\sin^2 na}{n^2}=\frac{\pi^2}{6}-\frac{a\pi-a^2}{2} \tag{26}$$

3.3.3 Series 3

The question in mind is that for $0<a<\pi$, prove that

$$\sum_{n=1}^{\infty}\left(-1\right)^{n-1}\frac{cosnx}{n}=ln\left(2cos\frac{x}{2}\right)\left(-\pi<x<\pi\right). \tag{27}$$

Expanding $ln\left(2cos\frac{x}{2}\right)$ over $\left(-\pi,\pi\right)$ as a Fourier series, it is calculated that the coefficients

$$\begin{aligned} a_0&=\frac{1}{\pi}\int_{-\pi}^{\pi}ln\left(2cos\frac{x}{2}\right)dx=2ln2+\frac{2}{\pi}\int_0^{\pi}lncos\frac{x}{2}dx\\ &=2ln2-\frac{4}{\pi}\int_0^{\frac{\pi}{2}}lncosxdx=0 \end{aligned} \tag{28}$$

$$\begin{aligned} a_n&=\frac{1}{\pi}\int_{-\pi}^{\pi}ln\left(2cos\frac{x}{2}\right)cosnxdx=\frac{2}{n\pi}\int_0^{\pi}ln\left(2cos\frac{x}{2}\right)d\sin nx\\ &=\frac{1}{n\pi}\int_0^{\pi}\frac{\sin nx\sin\left(\frac{x}{2}\right)}{cos\left(\frac{x}{2}\right)}dx\\ &=\frac{\left(-1\right)^{n-1}}{n\pi}\int_0^{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}coskx+\frac{1}{2}+\sum_{k=1}^{n-1}coskx\right)dx=\frac{\left(-1\right)^{n-1}}{n} \end{aligned} \tag{29}$$

In addition, $b_n=0$. Go one step further and replace x with $x+\pi$, there will be

$$\sum_{n=1}^{\infty}\frac{cosnx}{n}=ln\left(2\sin\frac{x}{2}\right)\left(0<x<2\pi\right). \tag{30}$$

After seeing the examples in Sec. 3.3, the reader should have a rudimentary understanding of more complex topics. Solving such problems requires proficiency in Fourier expansion.

4. Conclusion

In the previous part of the paper, the author discussed the basic content of Fourier series sum in detail. The author first gives the convergence theorem and Dini theorem, and then extends other reliable discriminant methods. After that, the Parseval equation is derived from the square mean convergence, and the term-by-term integral theorem of Fourier series is given. In the third part, the author lists several representative classical examples, p-series summation, staggered series summation and more complex and challenging examples. On the basis of Fourier series, it also introduces summation techniques from other angles, which is of great reference value and triggers readers to think. In the process of proof and calculation, the author still found many problems. For example, the point convergence of Fourier series is an extremely difficult problem, and it is difficult to make effective progress through simple methods. In the early days of research, scholars mostly cited examples of divergence, which is not an ordinary problem. At the same time, the authors also found that solving series through Fourier series actually depends very much on proficiency and skill. This requires the reader to be proficient in the Fourier expansion of the correlation function and the technique of function integration. This inspired researchers to further explore the solution of series sums from more angles and more general solution processes.

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