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# An evaluation of the significance of the Riemann Rearrangement Theorems on other algebraic theorems and mathematical concepts of infinity

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**Abstract:**

The aim of this dissertation is to investigate the Riemann Rearrangement Theorem and its connections to other mathematical theorems that are related to conditionally convergent series and concept of infinity.

It applies a comprehensive review of academic articles about the Riemann Rearrangement Theorem and its relationships with theorems such as Dirichlet's Theorem and Ohm's Rearrangement Theorem. Examples of numerical calculation and case studies are also analyzed to illustrate how these theorems influence one another.

The results show how important the Riemann Rearrangement Theorem is for comprehending the convergence of conditionally convergent series. It demonstrates how the Riemann Rearrangement Theorem allows the target sum to be attained by rearranging the terms in the series, and how Dirichlet's Theorem reinforces the absolutely convergent series' stability under such rearranging. The research demonstrates how Ohm's Theorem offers useful strategies for rearranging terms to get particular sum values. These connections shed light on the intricate interactions between these theorems, improving our understanding of how series behave when they converge.

This research emphasizes the importance of discovering the relationships between the Riemann Rearrangement Theorem and other algebraic theorems related to conditionally convergent series. By highlighting these connections, the dissertation suggests a comprehensive approach on applying these concepts and refers to a greater understanding of how different theorems can enhance our understanding of series and convergence in the field of mathematics.

**Keywords:** Riemann Rearrangement Theorem, Conditionally Convergent Series, Partial Sums, Infinite Series.

# 1. Introduction

What is the origin of mathematics? Throughout history, mathematics has developed significantly, from the simplest 1+1 to seven of the hardest mathematical problems in the world, that have yet to be solved. As mathematicians continued to tackle it, mathematics gradually evolved into many different categories, from real numbers to imaginary numbers, from equations to functions, from exponential to logarithms, and most importantly, from finite to infinite.

The notion of infinity is not only philosophical, but also plays a fundamental role in various mathematical theories and applications. Within the field of infinity, infinite series has intrigued and challenged mathematicians heavily throughout history. Since the ancient time when Zeno first sent Achilles chasing after the tortoise, infinite series have been a source of wonder and amusement because they can be manipulated to appear to contradict our understanding of numbers and nature (Galanor, 1987).

Mathematicians of the late seventeenth and eighteenth centuries were often confused by the result they would get while dealing with infinite series, and in the nineteenth century, it was discovered that the cause of these puzzles was often related to divergent series. “Divergent series are the invention of the devil”, said by Neils Hendrik Abel in a letter he wrote to his friend in 1826. By using them, one may draw any conclusion one pleases, and that is why these series have produced so many fallacies and so many paradoxes (Kline, 1972).

Many things could go wrong when discussing infinite series. For example, if we let $S$ be the sum of the alternating harmonic series, that is

$$S=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

What is wrong with $S$ in the equation above? Through some complex calculations, we can know that the sum of it is $ln2$. However, by simply rearranging the terms in the series, its sum can change from $S$ to $2S$. This is an absolutely magical phenomenon, as it is stating that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}$ might not be equal to $-\frac{1}{4}+\frac{1}{3}-\frac{1}{2}+1$. How is that even possible?

Let’s see how this magic happens. First, let’s list first few terms in this series:

$$(1)\,S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\frac{1}{13}-\frac{1}{14}+\frac{1}{15}-\dots$$

Then, multiply each term by 2:

$$(2)\,2S=2-1+\frac{2}{3}-\frac{1}{2}+\frac{2}{5}-\frac{1}{3}+\frac{2}{7}-\frac{1}{4}+\frac{2}{9}-\frac{1}{5}+\frac{2}{11}-\frac{1}{6}+\frac{2}{13}-\frac{1}{7}+\frac{2}{15}-\frac{1}{8}+\dots$$

Collect and cancel the terms with the same denominator, as the arrow shows:

$$(3)\,2S=2-1+\frac{2}{3}-\frac{1}{2}+\frac{2}{5}-\frac{1}{3}+\frac{2}{7}-\frac{1}{4}+\frac{2}{9}-\frac{1}{5}+\frac{2}{11}-\frac{1}{6}+\frac{2}{13}-\frac{1}{7}+\frac{2}{15}-\frac{1}{8}+\dots$$

Now we get this, which looks unbelievably similar:

$$(4)\,2S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\dots$$

We can see that on the right side of equation (1) and (4), that consist of the same terms starting the series. In other words: $S=2S$, which tells us that $1=2$.

This paradox where $1=2$ always happens when we try to solve problems and calculate with infinite series. This unique aspect of the mathematical concept of infinity has inspired thoughtful reflection and study, which has resulted in the development of a number of theories and theorems that attempt to explain its mysterious nature and implications.

The Riemann Rearrangement Theorem, sometimes referred to as the Riemann Series Theorem, is a fundamental mathematical idea put out by German mathematician Bernhard Riemann in the 19th century and is an essential element of this topic. According to this theorem, if an infinite series of real numbers is conditionally convergent, then there is a possibility to rearrange its terms in a way that causes the series to diverge or converge to a different sum.

Therefore, it is crucial to investigate this theorem considering the context of other algebraic theorems in order to deepen our understanding of infinity and its mathematical implications. These are the fundamental ideas that are explored further and studied using secondary methodology in the later sections of this dissertation.

# 2. Literature Review

The literature review provides an overview of the historical background of the Riemann Rearrangement Theorem. It discusses the concepts of convergence and divergence of infinite series, various convergence tests, Riemann Rearrangement Theorem, related theorems with proofs given in the appendix and some other numerical examples used.

Riemann’s rearrangement theorem demonstrates how the order of terms determines the convergence of a condition-

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ally convergent series. This idea highlights the relationship that exists between how terms are arranged in an infinite series and whether or not they eventually diverge or converge, showing the impact that order has on an infinite series' behavior. It is clear that the Riemann Rearrangement Theorem has applications rather than theoretical mathematics when one looks at its more general implications. This theorem's broad implications encompass several mathematical fields, such as algebra, and calculus, among others, highlighting the theorem's widespread influence on mathematics.

## 2.1 Historical background of Bernhard Riemann and his Rearrangement theory

Bernhard Riemann (17 September 1826 - 20 July 1866) was a German mathematician. In 1846, he attended the University of Gottingen to study theology as his father had encouraged him to do. Eventually, his father gave him permission to study mathematics under Moritz Stern and Carl Friedrich Gauss. In 1852, Riemann began working on the results of Dirichlet involving the Fourier series (Agana, 2015).

Dirichlet found that certain types of series could be rearranged to a sum different from the sum of the original series. Later, Riemann discovered that this works for any conditionally convergent series. This result became known as Riemann's Rearrangement Theorem in his Fourier series paper, "On the Representation of a Function by a Trigonometric Series," which he completed in 1853. However, his paper was not published until after his death (J. J. O'Connor and E. F. Robertson, 2009).

## 2.2 Convergence and divergence of infinite series

In terms of the limit of sequences, if $\lim_{n\to\infty} a_n$ exists and is finite, the sequence is convergent. If $\lim_{n\to\infty} a_n$ does not exist or is infinite, the sequence is divergent. The sequence diverges to $\infty$ if $\lim_{n\to\infty} a_n = \infty$ and if $\lim_{n\to\infty} a_n = -\infty$, the sequence diverges to $-\infty$. In terms of partial sums, If the sequence of partial sums is a convergent sequence (i.e. its limit exists and is finite) then the series is also called convergent and, in this case, if $\lim_{n\to\infty} S_n = S$ then, $\sum_{i=1}^{\infty} a_i = S$. Likewise, if the sequence of partial sums is a divergent sequence (i.e. its limit doesn't exist or is plus or minus infinity) then the series is also called divergent (Dawkins, 2022).

Example 2.2.1.Determine if the series

*Example*2.2.1.*Determineiftheseries*$\sum_{n=1}^{\infty} n$ is convergent or divergent.

*Proof*. The general formula for the partial sums is

$$S_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

To determine if the series is convergent, we need to see if the sequence of partial sums

$$\left\{\frac{n(n+1)}{2}\right\}_{n=1}^{\infty}$$

is convergent or divergent. The limit of the sequence terms is

$$\lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

Therefore, the sequence of partial sums diverges to $\infty$ and so the series also diverges.

Example 2.2.2.Determine if the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ is convergent or divergent.

*Proof*. In this section we are focused on the idea of convergence and divergence and so process for finding the formula is put off. The general formula for the partial sums is,

$$S_n = \sum_{i=2}^{n} \frac{1}{i^2-1} = \frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{3}{4} - \frac{1}{2n} - \frac{1}{2(n+1)}\right) = \frac{3}{4}$$

The sequence of partial sums converges and so the series converges also and its value is $\frac{3}{4}$.

## 2.3 Convergence tests of infinite series

Theoretically, we can determine the convergence of a series by examining its $n$ th partial sum, for example a geometric series. However, a formula for the $n$ th partial sum of most infinite series cannot be found (Galanor, 1987). Augustin-Louis Cauchy, as well as Abel and Dirichlet, realized this difficulty and was among the first to advise several theorems or tests to determine the convergence of a series (Riemann, 1876).

Theorem 2.3.1.The sum of two convergent series is a convergent series.If

$\sum a_n = S and \sum b_n = T then \sum (a_n + b_n) = S + T$.

Theorem 2.3.2.The sum of a convergent and a divergent series is a divergent series.

Theorem 2.3.3. $\sum_{1}^{\infty} a_n$ and $\sum_{i}^{\infty} a_n$ both converge or both diverge.In other words,the first finite number of terms do not determine the convergence of a series.

Theorem 2.3.4.If the series $\sum a_n$ convergens, then $\lim_{n\to\infty} a_n = 0$.

For a proof of this theorem, see Proof 8.1 in appendix.

Theorem 2.3.5.If $\sum |a_n|$ convergens, then $\sum a_n$ convergens.

Theorem 2.3.6.The compsrison test.If the series $\sum a_n$ and $\sum b_n$ have only positive terms with $a_n \leqslant b_n$ for all $ln \geqslant 1$, and

(1) if $\sum b_n$ converges,then $\sum a_n$ converges;

(2) if $\sum a_n$ diverges,then $\sum b_n$ diverges.

For a proof of this theorem, see Proof 8.2. in appendix.

Theorem2.3.7. Leibniz' alternating series test. The alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ convergnes if the sequence $\{a_n\}$ is monotone decreasing to 0. In other words, suppose we have a series $\sum a_n$ and either $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$ where $b_n \geqslant 0$ for. all $n$. If $\lim_{n\to\infty} b_n = 0$ and $\{b_n\}$ is a decreasing sequence, the series $\sum a_n$ is convergnent.

For a proof of this theorem, see *Proof* 8.3. in appendix.

Theorem2.3.8. Cauchy's integral test. Suppose that $f(x)$ is a continuous, positive. and decreasing function on the inetrval $[k, \infty)$ and that $f(n) = a_n$ then,

(1) if $\int_k^{\infty} f(x)dx$ is convergent, then $\sum_{n=k}^{\infty} a_n$ converges.

(2) if $\int_k^{\infty} f(x)dx$ is divergent, then $\sum_{n=k}^{\infty} a_n$ divereges.For a proof of this theorem, see *Proof* 8.4. in appendix.

## 2.4 Two Types of Convergent Infinite Series

There are two types of convergent series (Galanor, 1987):

A series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ converges.

A series $\sum a_n$ is called conditionally convergent if $\sum a_n$ converges but $\sum |a_n|$ diverges.

Example2.4.1. Determine if the alternating harmonic series $\sum \frac{(-1)^{n-1}}{n}$ is an. absolutely convergent series or conditionally convergent series.

*Proof*. The absolute values of the terms of this series are monotonic decreasing to 0. By theorem 2.3.7, Leibniz' alternating series test, we can conclude that the alternating harmonic series converges. If we take the absolute value of all the terms, we get the harmonic series, which, as we have seen, diverges, because $\lim_{n\to\infty}\frac{1}{n}$ doesn't exist. Hence the alternating harmonic series is a conditionally convergent series (Galanor, 1987).

## 2.5 Dirichlet's discovery about absolutely convergent series

German mathematician Johann Peter Gustav Lejeune Dirichlet came up with an important result involving the rearrangement of terms of certain series (Agana, 2015).

Dirichlet was the first to notice that terms in certain series could be rearranged to a sum different from the original series which was later founded by Bernhard Riemann that this was due to conditionally convergent series. In 1837, Dirichlet published a paper proving that the sum remains the same when rearranging terms in an absolutely convergent series (Gupta, S. L., and Nisha Rani, 1975).

Theorem2.5.1. If $\sum f_n$ is absolutely convergent, and converges to $\alpha$, then every . rearrangement of $\sum f_n$ also converges to $\alpha$.

## 2.6 Ohm's Rearrangement Theorem

In 1839, a German mathematician by the name of Martin Ohm came up with the following rearrangement theorem (Ohm, 1839).

Theorem2.6.1. For $p$ and $q$ positive integers, rearrange $\sum_{n\geqslant 1}^{\infty}\frac{(-1)^{n-1}}{n}$ by taking the first positive terms, then the first $q$ negative terms, then the next $p$ positive terms, and so on. The rearranged series converges to $\ln(2) + \frac{1}{2}\ln\left(\frac{p}{q}\right)$.

## 2.7 Riemann's Rearrangement Theorem

In 1852, German mathematician Bernhard Riemann came up with the explanation for Peter Lejeune Dirichlet's discovery that one can change the sum of a conditionally convergent series by rearranging its terms. His explanation was the following two theorems (Riemann, 1876).

Theorem 2.7.1. In a conditionally convergent series,the sum of the positive terms is a divergent series and the sum of the negative terms is a divergent series.

Theorem2.7.2.A series $\sum f_n$ is conditionally convergent if and only if for each real. number $\alpha$, there is a rear-

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rangement of $\sum f_n$ that converges to $\alpha$ .

### 2.8 Sierpiński's Theorem

After a well-known theorem of Riemann, the order of terms under any convergent series that is not absolutely convergent can always be modified so that the sum of the series is a value arbitrarily given in principle. It is possible to demonstrate his theorem by changing appropriately the relative frequency of positive and negative terms in the series given, as Sierpiński said. In a meeting on March 6, 1911, the following theorem is presented (Sierpiński, 1911).

Theorem2.8.1. Let $(f_n)$ be conditionally convergent where $U = \sum f_n$ , and let $V \neq U$ . be a real number. If $V > U$ , there exists an explicitly described rearrangement $\pi$ with the property that each positive term of $f_n$ is left in place (if $f_n > 0$ , then $\pi(n) = n$ ) and $\sum f_{\pi(n)} = V$ . If $V < U$ , there exists an explicitly described rearrangement $\pi$ with the property that each negative term of $f_n$ is left in place (if $f_n < 0$ , then $\pi(n) = n$ ). and $\sum f_{\pi(n)} = V$

### 2.9 Lévy-Steinitz Theorem

By Riemann's Theorem a conditionally convergent series of real numbers can be rearranged to sum to any $\alpha \in \mathbb{R}$. This idea can be extended to complex numbers and more generally, the Lévy-Steinitz Theorem gives a similar result in $n$ dimensions (Rosenthal, 1987).

Theorem2.9.1. Let $\sum T$ be a given series in $\mathbb{R}^n$ . Let $R$ be the set of all sum of . rearrangements of $\sum T$ in $\mathbb{R}^n$ .Then $R$ is either $\varnothing$ or a translate of a subplace (that is, $R = v + M$ for some vector $v$ and some linear subplace $M$)

### 2.10 Research Gap

In the numerous studies conducted on the Riemann Rearrangement Theorem (RRT), there is a conspicuous absence in consideration of its influence on algebraic theorems and mathematical concepts of infinity. In general, the theorem is predominantly covered and discussed in the field of analysis, but the amount of academic research that thoroughly examines the implications and applications of the theorem in algebraic structures and theory appears very limited. As a result, there is a distinct lack of literature on the potential connections and interactions between RRT and algebraic concepts, which greatly limits understanding of this powerful theorem. The RRT undoubtedly has weighty implications for series and convergence, which are widely documented in the literature. However, its influence and significance in algebraic structures are not, so further research is needed to elucidate this important aspect (Kline, 1972).

Moreover, the practical application of the RRT, particularly in solving mathematical problems relevant to physics, engineering, economics, and other disciplines, is certainly a promising area that has not been sufficiently explored, apparently. Additional research in this area holds the potential to shed light on the practical and real-world utility of RRT, thereby expanding its reach beyond theoretical frameworks to more practical and pragmatic applications. In order to place the theoretical status of RRT within the scope of practice, initial information should be gathered, relate it to real-world applications, and show that the theory is very valuable in the problems that it can solve. It seems that academic background itself may not be enough to discuss how these approaches imply fundamental solutions to real-life problems.

Aside from its practical applications, exploring the educational benefits of incorporating the RRT into an extended mathematics curriculum is equally pivotal. The rich theoretical connotation of RRT, coupled with the promising practical significance that has been proposed, shows that the theorem has the potential to strengthen the teaching and learning of other algebraic concepts. Understanding the impact of RRT on academic curricula and the potential it holds to inspire deeper mathematical thinking among students is an area that has not been thoroughly explored (Grattan-Guinness, 1970). As a result, a well-planned study evaluating the educational applications of RRT to create a solid understanding of mathematical concepts is required. Furthermore, investigating how RRT could promote critical thinking and problem-solving abilities could provide valuable insights into the larger implications of its implementation in mathematics education.

The lack of research on how to apply RRT to algebraic theorems, the practical application of RRT in numerous fields, and its educational value in mathematics curricula all suggest intriguing areas for future research and further development.

## 3. Methodology

The project is based exclusively on secondary research methodology. Data is gathered from a range of online relevant and credible sources including academic papers, articles published within higher education and is presented in both quantitative and qualitative format. The numerical data includes a range of relevant equations, complex calculations and graphs with supporting analytical discussion and explanation. Concepts and perspectives arising from the research form a significant element of the Results and Discussion section of the paper. A reading log is shown in the appendix with critical comment relating to the relevance and significance of sources used.

# 4. Results and Discussion

## 4.1 Antecedent theorems of Riemann's Rearrangement Theorem

Before Riemann's discovery of the conditionally convergent series, two earlier theorems that are closely related to it and affect Riemann's theorems significantly, as mentioned in the Literature Review section, are Lejeune Dirichlet's theorem on absolutely convergent series and Martin Ohm's rearrangement theorem. In this section 4.1, these two theorems are recalled again.

Theorem4.1.1.If $\sum f_n$ is absolutely convergent, and converges to $\alpha$ , then every. rearrangement of $\sum f_n$ also converges to $\alpha$ .

For a proof of this theorem, see Proof 8.5. in appendix.

Theorem4.1.2. For $p$ and $q$ positive integers, rearrange $\sum_{n\geqslant 1}^{\infty}\frac{(-1)^{n-1}}{n}$ by taking the first $p$ positive terms, then the first $q$ negative terms, then the next $p$ positive terms, and so on.The rearranged series converges to $\ln(2)+\frac{1}{2}\ln\left(\frac{p}{q}\right)$ .

For a proof of this theorem, see *Proof* 8.6. in appendix.

## 4.2 Mathematical derivation of Riemann's Rearrangement Theorem

Theorem4.2.1.In a conditionally convergent series, the sum of the positive terms is a divergent series and the sum of the negative terms is a divergent series.

*Proof*. First, a conditionally convergent series must have an infinite number of positive and negative terms. If all its terms were positive or all were negative, it would be an absolutely convergent series. If $\sum a_n$ has only a finite number of negative terms, then the remaining series of positive terms must converge, since by theorem 2.3.3, the first finite number of terms do not count when we determine the convergence or divergence of an infinite series. This result would mean that $\sum a_n$ is an absolutely convergent series. (Riemann, 1876)

We take a conditionally convergent series $\sum a_n$ and separate it into two infinite series, one of all the positive terms and the other of all the negative terms and represent these series by $\sum a_n^+$ and $\sum a_n^-$, respectively. So that we can successfully recover the original series without rearranging terms by writing $\sum a_n=\sum a_n^+ + \sum a_n^-$, we define the terms $a_n^+$ and $a_n^-$ as follows: $a_n^+=\begin{cases} a_n \leftarrow a_n>0 \\ 0\leftarrow a_n<0\end{cases}$ and

$$a_n^-=\begin{cases} 0\leftarrow a_n>0 \\ a_n\leftarrow a_n<0\end{cases}.$$

For the convergence of $\sum a_n^+$ and $\sum a_n^-$, four possibilities exist:

Case 1: $\sum a_n^+$ converges and $\sum a_n^-$ converges.

Case 2: $\sum a_n^+$ converges and $\sum a_n^-$ diverges.

Case 3: $\sum a_n^+$ diverges and $\sum a_n^-$ converges.

Case 4: $\sum a_n^+$ diverges and $\sum a_n^-$ diverges.

Using the definitions of absolutely and conditionally convergent series, Riemann showed that cases 1, 2, and 3 are impossible and hence, case 4 follows. His explantion is shown below:

We can't have case 1, for suppose

$$\sum a_n^+ = Sand \sum a_n^- = -T$$

With $T>0$, then

$$\sum |a_n^-| = T$$

In this case, since

$$\sum |a_n| = \sum a_n^+ + \sum |a_n^-|$$

$\sum |a_n|$ is the sum of two convergent series. Therefore by theorem 2.3.1, $\sum |a_n|$ must converges to $S+T$. This result means that $\sum a_n$ would be an absolutely convergent series, not contionally convergent as required.

Case 2 is not valid, because if we add both the convergent series $\sum a_n^+$ and the divergent series $\sum a_n^-$, the resulting series $\sum a_n$ will diverge, as theorem 2.3.2 claimed. However, we know that $\sum a_n$ is conditionally convergent and hence must converge. Case 3 is essentially the same as case 2. Therefore, we must have case 4, $\sum a_n^+$ and $\sum a_n^-$ are both divergent series for a conditionally convergent series. The divergence of the two series is the key idea in proving the second part of Riemann's rearrangement theorem. It offers an insight as to why the sum of a conditionally convergent series can be changed by rearranging terms. In fact, as we will now see, the terms can be rearranged to add up to any number we wish!

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Theorem4.2.2.A series $\sum f_n$ is conditionally convergent if and only if for each real. number $\alpha$, there is a rearrangement of $\sum f_n$ that converges to $\alpha$.

*Proof*. ($\Leftarrow$) Note that this follows from Dirichlet's Theorem 2.4.1, which states that an absolutely convergent series converges to the same value no matter how it is rearranged.

($\Rightarrow$) Suppose $\sum f_n$ is conditionally convergent. We want to show there is a rearrangement, $\left(f_{\pi(n)}\right)$, of $\left(f_n\right)$ whose series converges to the real number $\alpha$. (Riemann, 1876)

First, consider the subsequence of positive terms of $\left(f_n\right)$, call it $\left(a_n\right)$, and the subsequence of negative terms of $\left(f_n\right)$, call it $\left(b_n\right)$. Then by Theorem 2.5.2, $\sum a_n = \infty$ and $\sum b_n = -\infty$.

We know that $\sum a_n = \infty$. This implies that there exists some natural number $N$ such that

$$\sum_{k=1}^{N} a_k > a$$

Now let $N_1 = N$ be the least such number, and consider the partial sum $S_1 = \sum_{k=1}^{N_1} a_k$, so,

$$S_1 = \sum_{k=1}^{N_1} a_k > \alpha$$

$$S_1 = \sum_{k=1}^{N_1 - 1} a_k \leqslant \alpha$$

$$0 < S_1 - \alpha \leqslant a_{N1}$$

Now to $S_1$, add just enough terms from $\left(b_1\right)$ in order so that the resulting partial sum

$$\sum_{k=1}^{N_1} a_k + \sum_{i=1}^{M} b_i$$

is now less than or equal to $\alpha$. Note that this is possible since $\sum b_i = -\infty$.

Letting $M_1$ be the least such number $M$, and setting

$$S_2 = \sum_{k=1}^{N_1} a_k + \sum_{i=1}^{M_1} b_i$$

$$0 \leqslant \alpha - S_2 < -b_{M1}$$

Continuing this process, we get partial sums that alternate between being larger and smaller than $\alpha$, and each time choosing the next smallest $N_k$ or $M_k$, we get the following rearrangement for $\left(f_n\right)$,

$$a_1, a_2, \cdots, a_{N_1}, b_1, b_2, \cdots, b_{M_1}, a_{N_1+1}, \cdots, a_{N_2}, b_{M_1+1}, \cdots, b_{M_2}, a_{N_2+1}, \cdots$$

Note that for all odd $i$, we have that $\left|S_i - \alpha\right| \leqslant a_{N_i}$, and for all even $j$, we have that $\left|S_j - \alpha\right| \leqslant -b_{M_j}$. Now for any $n$

$$S_{2n+1} > S_{2n+1} + b_{M_n+1} > S_{2n+1} + b_{M_n+1} + b_{M_n+2} > \cdots > S_{2n+2} - b_{M_n+1} > \alpha$$

And

$$S_{2n+2} < S_{2n+2} + a_{N_{n+1}+1} < S_{2n+2} + a_{N_{n+1}+2} < \cdots < S_{2n+3} - a_{N_{n+2}} \leqslant \alpha$$

So the partial sums of rearrangement $\left(f_{\pi(n)}\right)$ between $S_{2n+1}$ and $S_{2n+2} - b_{M_n+1}$ are bounded between $\alpha$ and $a_{N_{n+1}}$, and all partial sums between $S_{2n+2}$ and $S_{2n+2} - b_{M_n+1}$ are bounded between $\alpha - b_{M_n+1}$ and $\alpha$.

Since $\sum f_n$ converges, we notice that $\left(f_n\right)$ converges to 0. Therefore, $\left(a_{N_i}\right)$ and $\left(b_{M_j}\right)$ also converge to 0. Hence, the partial sums of $\left(f_{\pi(n)}\right)$ converge to $\alpha$; that is, $\sum f_{\pi(n)} = \alpha$, as required.

Example 4.2.1.Consider the usual Alternating Harmonic Series

$$\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

We already know that this sum converges to $\ln(2)$. However, let us show that this is true by applying Riemann's Rearrangement Theorem. So first consider the series of positive terms of $\sum \frac{(-1)^{n-1}}{n}$, call it $\sum a_n$, and the series of negative terms of $\sum \frac{(-1)^{n-1}}{n}$, call it $\sum b_n$.

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Now using Riemann's Rearrangement Theorem, we get the partial sum $S_1$ by adding . just enough terms from $\sum a_n$ so that $S_1 > \ln(2) \approx 0.6931$.

$$S_1 = 1 > ln(2)$$

To get $S_2$, we add just enough terms from $\sum b_n$ to $S_1$ so that $S_2 < \ln(2)$.

$$S_2 = 1 - \frac{1}{2} = \frac{1}{2} < ln(2)$$

To get $S_3$, we again add just enough terms from $\sum a_n$ to $S_2$ so that $S_3 > \ln(2)$.

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3} \approx 0.8333 > ln(2)$$

And likewise, we add just enough terms from $\sum b_n$ to $S_3$ to get $S_4$ so that $S_4 < \ln(2)$ .

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx 0.5833 < ln(2)$$

Continuing this process,we get the following partial sums:

$$S_{10} \approx 0.6456 < ln(2)$$
$$S_{21} \approx 0.7164 > ln(2)$$
$$S_{30} \approx 0.6768 < ln(2)$$
$$S_{41} \approx 0.7052 > ln(2)$$
$$S_{50} \approx 0.6832 < ln(2)$$

Each time we add just enough negative terms to get the new corresponding patial sum. so that it is less than $\ln(2)$ , the sums increase and approach the sum of the Alternating. Harmonic Series. Likewise, each time we add just enough positive terms to get the new. partial sum, the partial sums decrease and approach $\ln(2)$ . In other words, as $k$ . increases, for all $k \in \mathbb{N}, S_{2k} \to \ln(2)^-$(from the left) , and as $k$ increases, $S_{2k+1} \to \ln(2)^+$ . (from the right).

### 4.3 Later forms and generalizations of Riemann's Rearrangement Theorem

After Riemann's Rearrangement Theorem about the conditionally convergent series, there were two later theorems or generalizations that are closely related to it and got affected by Riemann's theorems significantly, as mentioned in the Literature Review section, are Sierpiński's Theorem focusing on changing the relative frequency of the way rearranging terms and Lévy-Steinitz Theorem extending the idea into complex number and different dimensions. In this section 4.3, these two theorems are recalled again with their proof outlined.

Theorem4.3.1. Let $(f_n)$ be conditionally convergent where $U = \sum f_n$ , and let $V \neq U$ . be a real number. If $V > U$ , there exist an explicitly described rearrangement $\pi$ with the property that each negative term of $f_n$ is left in place (if $f_n < 0$ , then $\pi(n) = n$ ). and $\sum f_{\pi(n)} = V$ . If $V < U$ , there exists an explicitly described rearrangement $\pi$ with.

the property that each negative term of $ f_{n} $ is left in place (if $f_n < 0$ , then $\pi(n) = n$ ). and $\sum f_{\pi(n)} = V$

For a proof of this theorem, see *Proof* 8.7. in appendix.

Theorem4.3.2. Let $\sum T$ be a given series in $\mathbb{R}^n$ . Let $R$ be the set of all sum of rearrangements of $\sum T$ in $\mathbb{R}^n$ . Then $R$ is either emptyset or a translate of a subplace (that is, $R = v + M$ for some vector $v$ and some linear subplace M). (Rosenthal,1987).

For a proof of this theorem, see *Proof* 8.8. in appendix.

### 4.4 The effects of antecedent theorems on Riemann's Rearrangement Theorem

Dirichlet's theorem about absolute convergent series and Ohm's rearrangement theorem have significant influences on the Riemann Rearrangement Theorem, which is a crucial theorem in mathematical concepts of infinity. Dirichlet's theorem states that if a series is absolutely convergent and converges to a certain value, then every rearrangement of the series will also converge to the same value. Ohm's rearrangement theorem, on the other hand, provides a specific method for rearrangement of a series by alternating positive and negative terms, resulting in a convergent series with a specific sum.

Dirichlet's theorem and Ohm's rearrangement theorem have a significant impact on the Riemann rearrangement theorem. The Riemann Rearrangement Theorem claims that the terms in a conditionally convergent series can be rearranged to converge to any desired real number. Dirichlet's theorem provides a fundamental understanding of the convergence characteristics and stability of absolutely convergent series, which serve as the foundation for the Riemann Rearrangement Theorem.

In contrast, Ohm's rearrangement theorem provides insight into the specific strategies for rearranging series to achieve desired convergence. Ohm's approach of alternating positive and negative terms when rearranging a series and calculating a specific sum demonstrates the ability of controlling series to converge to varied values based on rearrangement patterns. Although Ohm's theorem focuses on a specific way for rearranging series, it helps us understand how rearrangements can alter the properties of series.

Example4.4.1. Consider again the Alternating Harmonic Series.Similar to Example .

4.2.1., we want to show that there exists a rearrangement that converges to $\alpha = \frac{3}{2}\ln(2) \approx 1.0397$ . Applying the formula of Riemann's Rearrangement Theorem we get. the following partial sums:

$$S_1 = 1 + \frac{1}{3} \approx 1.3333 > \frac{3}{2} ln(2)$$
$$S_2 = 1 + \frac{1}{3} - \frac{1}{2} \approx 0.8333 < \frac{3}{2} ln(2)$$
$$S_3 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} \approx 1.1762 > \frac{3}{2} ln(2)$$

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$$S_4 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} \approx 0.926190476 < \frac{3}{2} ln(2)$$

$$S_5 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} \approx 1.2191 > \frac{3}{2} ln(2)$$

Recall that for every odd partial sum, we are adding just enough positive terms not already used to make the partial sum larger than $\frac{3}{2}\ln(2)$ , and for every even partia sum, we are adding just enough negative terms not already used so that the partial sum is less than $\frac{3}{2}\ln(2)$ . Continuing, we get the following partial sums:

$$S_6 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} \approx 0.9615 < \frac{3}{2} ln(2)$$

$$S_7 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} \approx 1.1051 > \frac{3}{2} ln(2)$$

$$S_8 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} \approx 0.9801 < \frac{3}{2} ln(2)$$

$$S_9 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \frac{1}{17} + \frac{1}{19} \approx 1.0916 > \frac{3}{2} ln(2)$$

$$S_{10} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots + \frac{1}{17} + \frac{1}{19} - \frac{1}{10} \approx 0.9916 < \frac{3}{2} ln(2)$$

$$S_{35} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \cdots - \frac{1}{34} + \frac{1}{69} + \frac{1}{71} \approx 1.0538 > \frac{3}{2} ln(2)$$

An obvious pattern is found: for every odd partial sum, we need to add the first two positive terms not already used from $\sum a_n$ so that the partial sum is greater than. $\frac{3}{2}\ln(2)$ , and for each even partial sum, the first negative term not already used from.

$\sum b_n$ so that the partial sum is less than $\frac{3}{2}\ln(2)$ .

Notice that if we continue this process, the partial sums eventually converge to .

$\frac{3}{2}\ln(2).$

$$Thus, S_2 \leqslant S_4 \leqslant S_6 \leqslant S_8 \leqslant \cdots \leqslant \frac{3}{2}\ln(2) \leqslant \cdots \leqslant S_7 \leqslant S_5 \leqslant S_3 \leqslant S_1$$

So the rearrangement

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \cdots - \cdots - \frac{1}{34} + \frac{1}{69} + \frac{1}{71} + \cdots = \frac{3}{2}\ln(2) .$$

Recall Ohm's Rearrangement Theorem and consider A(2,1).Using Ohm's Theorem to . look at the partial sums \$ A(2,1) \$, we get the following first five partial sums: $C_1 = 1 + \frac{1}{3} + \left(-\frac{1}{2}\right) \approx 0.83 = S_2 < \frac{3}{2} ln(2)$

$$C_2 = 1 + \frac{1}{3} + \left(-\frac{1}{2}\right) + \left[\frac{1}{5} + \frac{1}{7} + \left(-\frac{1}{4}\right)\right] \approx 0.93 = S_4 < \frac{3}{2} ln(2)$$

$$C_3 = 1 + \frac{1}{3} + \left(-\frac{1}{2}\right) + \frac{1}{5} + \frac{1}{7} + \left(-\frac{1}{4}\right) + \left[\frac{1}{9} + \frac{1}{11} + \left(-\frac{1}{6}\right)\right] \approx 0.96 = S_6 < \frac{3}{2} ln(2)$$

$$C_4 = 1 + \frac{1}{3} + \left(-\frac{1}{2}\right) + \frac{1}{5} + \frac{1}{7} + \left(-\frac{1}{4}\right) + \cdots + \left[\frac{1}{13} + \frac{1}{15} + \left(-\frac{1}{8}\right)\right] \approx 0.98 = S_8 < \frac{3}{2} ln(2)$$

$$C_5 = 1 + \frac{1}{3} + \left(-\frac{1}{2}\right) + \frac{1}{5} + \frac{1}{7} + \left(-\frac{1}{4}\right) + \cdots + \left[\frac{1}{17} + \frac{1}{19} + \left(-\frac{1}{10}\right)\right] \approx 0.99 = S_{10} < \frac{3}{2} ln(2)$$

Observe that each partial sum $C_n$ from Ohm's Theorem is equal to each even partial. sum $S_{2n}$ from Riemann's Rearrangement Theorem, and we know that by Ohm's . Rearrangement Theorem, $C_n$ converges to $\frac{3}{2}\ln(2)$ , which aligns with Riemann's . Rearrangement Theorem.

## 4.5 The effects of Riemann's Rearrangement Theorem on later forms and generalization

Riemann's rearrangement theorem is important because it has far-reaching implications for series convergence and the relationship between rearrangement and target sum values. This theorem challenges the traditional understanding of series convergence by demonstrating that rearranging the terms of a conditionally convergent series can produce various sums, which can be changed to any desired real value. This flexibility in rearranging the terms provides an unusual viewpoint on series convergence, emphasizing the complicated relationship between the terms' summation order and the sum. The theorem has had a significant impact on the study of infinite series, particularly in the analysis of convergence and divergence, and has encouraged additional research into infinite series convergence.

Sierpiński's Theorem is based on the Riemann Rearrangement Theorem and offers clear conditions for rearranging terms within a conditionally convergent series to achieve specific target sums. Sierpiński's Theorem offers a systematic approach to understanding the possibilities and limitations of rearrangements in series convergence. It provides a precise description for rearranging terms without changing the relative order of negative terms. This theorem not only expands our knowledge of rearrangement strategies, but it also points out the significance of evaluating the sign and magnitude of terms as dealing with series.

The Lévy-Steinitz Theorem extends the idea into higher dimensions, specifically in $\mathbb{R}^n$, signifies a huge development in the application of rearrangement concepts to complex mathematical spaces. This theorem broadens the application of rearrangement theory beyond typical one-dimensional series by investigating rearrangements under multidimensional environment and establishes an interaction between the set of rearrangement sums and linear subspace. The influence of this theorem runs through algebraic theorems involving multidimensional rearrangements, providing new perspectives on rearrangement strategies and convergence properties in complex mathematical landscapes.

## 5. Evaluation

The content of the dissertation could be considered limited given the complexity of the subject matter and the author's stage in education at the time of writing. The level of complexity, knowledge and understanding extends way beyond the taught A-level curriculum of the author. Careful selection of calculations and mathematical notations consistent with the author's skill level and understanding ensure effective analysis and clarity of explanation.

It could be argued that a strength of the research is that two antecedent theorems were investigated, and two later forms and generalizations analyzed which provides further insight into the process and impact of the Riemann Rearrangement Theorems.

The Results and Discussion section shows some examples which are used as evidence for the later element of this section, which demonstrates an analysis and explanations of how Riemann Rearrangement Theorems were affected by earlier theorems and in turn, affected later forms and generalizations. The structure of this content is logical and aids accessibility for the reader.

Secondary sources supporting this research are credible, well established and widely accepted.

## 6. Conclusion

The Riemann Rearrangement Theorem holds significant importance in the topic of rearrangement theorems and mathematical concepts of infinity, with profound implications for various mathematical theorems. Its connections to antecedent theorems, such as Dirichlet's theorem and Ohm's rearrangement theorem shows its fundamental role in shaping our understanding of the convergent properties of infinite series and the rearrangement strategy. Dirichlet's theorem is well-known for ensuring convergent stability when rearranging absolutely convergent series, and it serves as the fundamental basis for Riemann's Rearrangement Theorem, highlighting that it converges identically regardless of rearrangement. The Ohm rearrangement theorem focuses on specific rearrangement approaches, demonstrating a range of convergence outcomes that can be reached using different rearrangement patterns.

The Riemann Rearrangement Theorem continues to impact later forms and generalizations, such as Sierpiński's Theorem and the extension of the Lévy-Steinitz Theorem to higher dimensions. Sierpiński's Theorem expands on the Riemann Rearrangement Theorem by introducing straightforward requirements for rearranging terms within conditionally convergent series to obtain targeted sums. This enhances our understanding of rearrangement strategies and their impact on series convergence. This method emphasizes the significance of evaluating the signs and magnitudes of terms in series operations, which influences later algebraic theorems dealing with rearrangement techniques and convergence patterns. The extension of Lévy-Steinitz Theorems to higher dimensions represents a big step forward in extending the concept of rearrangement to complicated mathematical spaces, moving beyond typical one-dimensional series to investigate multidimensional rearrangement problems in depth.

In conlcusion, the Riemann Rearrangement Theorem serves as a foundation for evaluating the significance of rearrangement theorems to mathematical concepts of infinity, as it discusses convergence properties, rearrangement strategies, and the interaction between series operations and convergence values in terms of infinity. Its long-lasting influence on later theorems and generalisations highlighted its importance in widening our comprehension of the concept of infinity through numerical calculation, paving the way for further research and development in the field of mathematical analysis.

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## 8. Appendix

*Proof* 8.1. *forTheorem*2.3.4.

*Proof*. Suppose that the series starts at $n=1$. If not, we could modify the proof below to meet the new starting place or we could do anindex shift to get the series to start at $n=1$. Then the partial sums are,

$$S_{n-1}=\sum_{i=1}^{n-1}a_i=a_1+a_2+a_3+a_4+\cdots+a_{n-1}$$

$$S_n=\sum_{i=1}^{n}a_i=a_1+a_2+a_3+a_4+\cdots+a_{n-1}+a_n$$

Use two partial sums to write,

$$a_n=S_n-S_{n-1}$$

We know that $\sum a_n$ is convergent, the sequence $\{S_n\}_{n=1}^{\infty}$ is also convergent and that $\lim_{n\to\infty}S_n=S$ for some finite value $S$. However, since $n-1\to\infty$ as $n\to\infty$ we also have $\lim_{n\to\infty}S_{n-1}=S$. We now have,

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}(S_n-S_{n-1})=\lim_{n\to\infty}S_n-\lim_{n\to\infty}S_{n-1}=S-S=0$$

?

*Proof* 8.2. *forTheorem*2.3.6.

*Proof*. Suppose that the series starts at $n=1$. If not, we could modify the proof below to meet the new starting place or we could do anindex shift to get the series to start at $n=1$.

Then the partial sums for each series are,

$$S_n=\sum_{i=1}^{n}a_i and T_n=\sum_{i=1}^{n}b_i$$

Because $a_n,b_n\geqslant 0$ so that,

$$S_n\leqslant S_n+a_{n+1}=\sum_{i=1}^{n}a_i+a_{n+1}=\sum_{i=1}^{n+1}a_i=S_{n+1}\Rightarrow S_n\leqslant S_{n+1}$$

$$T_n\leqslant T_n+b_{n+1}=\sum_{i=1}^{n}b_i+b_{n+1}=\sum_{i=1}^{n+1}b_i=T_{n+1}\Rightarrow T_n\leqslant T_{n+1}$$

So, both partial sums form increasing series.

Also, because $a_n\leqslant b_n$ for all $n$ we know that we must have $S_n\leqslant T_n$ for all $n$.

With these preliminary facts above we can start the proof of the test.

(1) Assume that $\sum_{n=1}^{\infty}b_n$ is a convergent series. Since $b_n\geqslant 0$ we know that,

$$T_n=\sum_{i=1}^{n}b_i\leqslant\sum_{i=1}^{\infty}b_i$$

However, $S_n\leqslant T_n$ for all $n$ and so for all $n$ we have,

$$S_n\leqslant\sum_{i=1}^{\infty}b_i$$

Finally, since $\sum_{n=1}^{\infty}b_n$ is a convergent series, it must have a finite value and so the partial sums, $S_n$ are bounded above, meaning there exists a number $m$ such that $S_n\leqslant m$ for every $n$. As a monotonic and bounded sequence is convergent, $\{S_n\}_{n=1}^{\infty}$ is a convergent series and so $\sum_{n=1}^{\infty}a_n$ is convergent.

(2) Assume that $\sum_{n=1}^{\infty}a_n$ is a divergent series. Since $a_n\geqslant 0$ we know that we must have $S_n\to\infty$ as $n\to\infty$. However, we also know that for all $n$ we have $S_n\leqslant T_n$ and therefore we can know that $T_n\to\infty$ as $n\to\infty$. Hence, $\{T_n\}_{n=1}^{\infty}$ is a

divergent series and so $\sum_{n=1}^{\infty} b_n$ is divergent.

?

*Proof* 8.3. *forTheorem*2.3.7.

*Proof*. Suppose that the series starts at $n=1$. If not, we could modify the proof below to meet the new starting place or we could do anindex shift to get the series to start at $n=1$.

First notice that because the terms of the sequence are decreasing for any two successive terms we can say,

$b_n - b_{n+1} \geqslant 0$

For even partial sums,

$S_2 = b_1 - b_2 \geqslant 0$

$S_4 = b_1 - b_2 + b_3 - b_4 = S_2 + b_3 - b_4 \geqslant S_2 because b_3 - b_4 \geqslant 0$

$S_6 = S_4 + b_5 - b_6 \geqslant S_4 because b_5 - b_6 \geqslant 0$

?

$S_{2n} = S_{2n-2} + b_{2n-1} - b_{2n} \geqslant S_{2n-2} because b_{2n-1} - b_{2n} \geqslant 0$

So, $\{S_{2n}\}$ is an increasing sequence.

Next, the general term can be written as,

$S_{2n} = b_1 - b_2 + b_3 - b_4 + b_5 + \cdots - b_{2n-2} + b_{2n-1} - b_{2n}$

$S_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) + \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$

Each of the quantities in parenthesis are positive and by assumption we know that $b_{2n}$ is also positive. So, this tells us that $S_{2n} \leqslant b_1$ for all $n$. We know that $\{S_{2n}\}$ is an increasing sequence that is bounded above and so we know that is must also converge.

Assume that the limit is $S$ or,

$\lim_{n\to\infty} S_{2n} = S$

Next, we can determine the limit of the sequence of odd partial sums, $\{S_{2n+1}\}$, as follows,

$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty}(S_{2n} + b_{2n+1}) = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} b_{2n+1} = S + 0 = S$

Hence, we know that both $\{S_{2n}\}$ and $\{S_{2n+1}\}$ are convergent sequences and they both have the same limit and so we can also know that $\{S_n\}$ is a convergent series with a limit of $S$. This in turn tells us that $\sum a_n$ is convergent (Apostol, 1967).

?

*Proof* 8.4. *forTheorem*2.3.8.

*Proof*. For the sake of proof, this will be working with the series $\sum_{n=1}^{\infty} a_n$. The original test statement was for a series that started at a general $n=k$, but the proof will be easier if we assume that that the series starts at $n=1$.

Another way of dealing with $n=k$ is we could do an index shift and start the series at $n=1$ and then do the Integral Test. Either way proving the test for $n=1$ will be sufficient. Also note that while we allowed for the first few terms of the series to increase or be negative in working problems this proof does require that all the terms be decreasing and positive.

Let's start off and estimate the area under the curve on the interval $[1, n]$ and we'll under- estimate the area by taking rectangles of width one and whose height is the right endpoint.

This gives the following figure. The image is from the e-book *CalculusII* by Paul Dawkins.

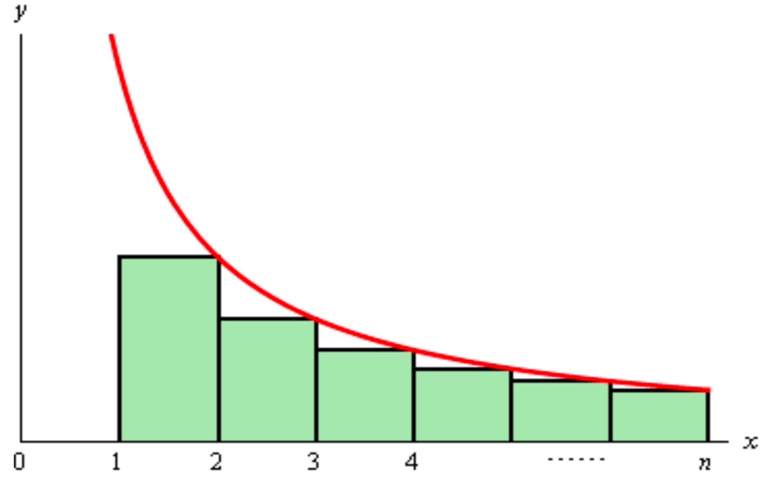


**Figure. 1**

Now, note that,

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$f(2)=a_2$
$f(3)=a_3$
?
$f(n)=a_n$
Then the approximate area is then,
$A \approx (1)f(2)+(1)f(3)+\cdots+(1)f(n)=a_2+a_3+a_4+\cdots+a_n$
And we know that this underestimates the actual area so,
$\sum_{i=2}^{n} a_i = a_2+a_3+\cdots+a_n < \int_1^n f(x)dx$
Now, let's suppose that $\int_1^\infty f(x)dx$ is convergent and so $\int_1^\infty f(x)dx$ must have a finite value. Also, because $f(x)$ is positive we know that,
$\int_1^n f(x)dx < \int_1^\infty f(x)dx$
This in turn means that,
$\sum_{i=2}^{n} a_i < \int_1^n f(x)dx < \int_1^\infty f(x)dx$
Our series starts at $n=1$ so this isn't quite what we need. However, it is easy to deal with.
$\sum_{i=1}^{n} a_i = a_1 + \sum_{i=2}^{n} a_i < a_1 + \int_1^\infty f(x)dx = M$
Now, we know that the sequence of partial sums, $S_n = \sum_{i=1}^{n} a_i$ are bounded above by $M$.
Next, because the terms are positive, we know that,
$S_n \leqslant S_n + a_{n+1} = \sum_{i=1}^{n} a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = S_{n+1} \Rightarrow S_n \leqslant S_{n+1}$
The sequence $\{S_n\}_{n=1}^{\infty}$ is also an increasing sequence. By this, we can know that the series of the partial sums $\{S_n\}_{n=1}^{\infty}$ converges and hence our series $\sum_{n=1}^{\infty} a_n$ is convergent.
The first part of the test is proven. This time let's overestimate the area under the curve by using the left endpoints of interval for the height of the rectangles as shown below. The image is from the e-book *CalculusII* by Paul Dawkins.

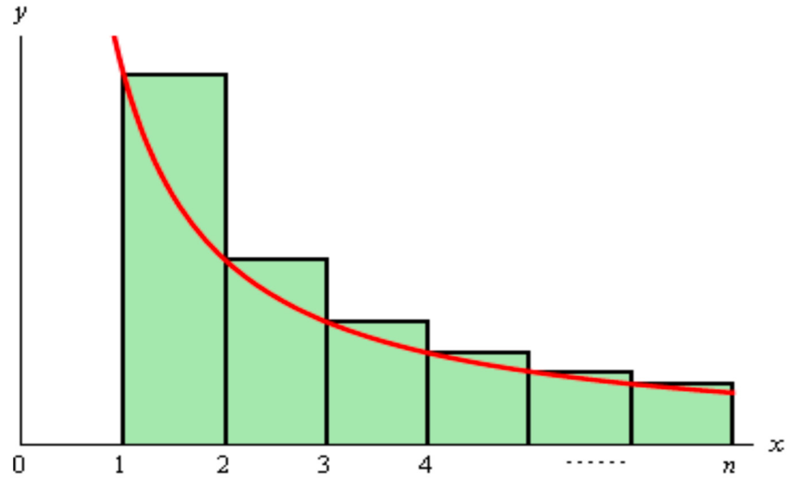


**Figure. 2**

In this case, the area is approximately,
$A \approx (1)f(1)+(1)f(2)+\cdots+(1)f(n-1)=a_2+a_3+a_4+\cdots+a_{n-1}$
Since we know this overestimates the area, we also then know that,
$S_{n-1} = \sum_{i=1}^{n-1} a_i = a_1+a_2+\cdots+a_{n-1} > \int_1^{n-1} f(x)dx$
Now, suppose that $\int_1^\infty f(x)dx$ is divergent. In this case this means that $\int_1^n f(x)dx \to \infty$ as $n \to \infty$ because $f(x) \geqslant 0$.
However, because $n-1 \to \infty$ as $n \to \infty$, we also know that $\int_1^{n-1} f(x)dx \to \infty$.
Therefore, since $S_{n-1} > \int_1^{n-1} f(x)dx$ we know that as $n \to \infty$ we must have $S_{n-1} \to \infty$.
This in turn tells us that $S_n \to \infty$ as $n \to \infty$. So, we now know that the series of partial sums $\{S_n\}_{n=1}^{\infty}$ is a divergent

series and so $\sum_{n=1}^{\infty} a_n$ is a divergent series (Dawkins, 2022).

*Proof* 8.5. *forTheorem*4.1.1.

*Proof*. Assume $\sum |f_n| < \infty$, and that $\sum f_n = \alpha$. Now let $(f_{\pi(n)})$ be a rearrangement of $(f_n)$. We need to show that: (Gupta, S. L., and Nisha Rani, 1975)

$$\sum_{n=1}^{\infty} f_{\pi(n)} = \alpha$$

Fix $\epsilon > 0$. Find $N$ such that for all $n > N$,

$$\left|\sum_{k=1}^{n} f_{\pi(k)} - \alpha\right| < \epsilon$$

Find $N_1$ such that for every $n \geqslant N_1$,

$$\left|\sum_{k=1}^{n} f_k - \alpha\right| < \frac{\epsilon}{2}$$

Find $N_2$ such that for every $n > N_2$,

$$\sum_{k=n}^{\infty} |f_k| < \frac{\epsilon}{2}$$

We may assume that $N_2 \geqslant N_1$.

Let there be some $N_3$ large enough so that $\{1,2,3,\cdots,N_2\} \subseteq \{\pi(1),\pi(2),\pi(3),\cdots,\pi(N_2)\}$. We claim that $N = N_3$ works. To see this, note that if $n > N_3$, then

$$\sum_{k=1}^{n} f_{\pi(k)} = \sum_{k=1}^{N_2} f_k + \sum_{j\in A} f_j$$

Where $A = \{\pi(1),\pi(2),\pi(3),\cdots,\pi(n)\} \setminus \{1,2,3,\cdots,N_2\}$. Therefore,

$$\left|\sum_{k=1}^{n} f_{\pi(k)} - \alpha\right| \leqslant \left|\sum_{k=1}^{N_2} f_k - \alpha\right| + \left|\sum_{j\in A} f_j\right|$$

$$\left|\sum_{k=1}^{n} f_{\pi(k)} - \alpha\right| < \frac{\epsilon}{2} + \sum_{j\in A} |f_j|$$

$$\left|\sum_{k=1}^{n} f_{\pi(k)} - \alpha\right| \leqslant \frac{\epsilon}{2} + \sum_{j=N_2+1}^{\infty} |f_j|$$

$$\left|\sum_{k=1}^{n} f_{\pi(k)} - \alpha\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

*Proof* 8.6. *forTheorem*4.1.2.

Denote by $A(p,q)$ the series resulting from this rearrangement. Let $S_k$ be the partial sums of $A(p,q)$ where specifically $S_{k}\text{i}^{-} = S_{k}\text{i}^{-}(p,q)$ denotes the $k^{th}$ partial sum of $A(p,q)$, and let $C_n = C_n(p,q)$ be the partial sum of $A(p,q)$, which is the result in $S$ from precisely adding $n$ blocks, where each block consists of $p$ positive terms and $q$ negative terms, consecutively.

Denote $S_n$ to be the partial sums of the usual alternating series, where $n$ denotes the terms. Call $C_n$ as $C_n = S_{f(n)} + R_n$ for some $f(n)$ and some "remainder" $R_n$. This breaks down the proof of Theorem 4.1.2 into the following:

(1) Verifying the explicit formula of $C_n(p,q)$ for each of the three types of rearrangements:

(a) $p = q$,

(b) $p > q$, and

(c) $p < q$.

(2) Stating that $S_n$ and $C_n$ converge to $ln(2) + \frac{1}{2} ln\left(\frac{p}{q}\right)$

General Case (a): $A(p,q), p = q$.

We now proceed with the more general case:

*Theorem*4.1.2.1.*Forpositiveintegerp*, $C_n(p,p) = S_{2pn}$ *foralln.*

*Proof*. Consider $A(p,p)$. We could argue that: (Ohm, 1839)

$C_n = C_n(p,p) = S_{2pn}$

By induction, note that:

$$C_1 = \left(1 + \frac{1}{3} + \ldots + \frac{1}{2p-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2p}\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2p-1} - \frac{1}{2p} = S_{2p}$$

$$C_2 = C_1 + \frac{1}{2p+1} - \frac{1}{2p+2} + \frac{1}{2p+3} - \frac{1}{2p+4} + \ldots + \frac{1}{4p-1} - \frac{1}{4p} = S_{4p}$$

Now suppose that:

$$C_n(p,p) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2pn-1} + \frac{1}{2pn} = S_{2pn}$$

By the inductive hypothesis,

$$C_{n+1}(p,p) = C_n(p,p) + \left(\frac{1}{2pn+1} + \frac{1}{2pn+3} + \ldots + \frac{1}{2pn+2p-1}\right) - \left(\frac{1}{2pn+2} + \frac{1}{2pn+4} + \ldots + \frac{1}{2pn+2p}\right)$$

$$= C_n(p,p) + \frac{1}{2pn+1} - \frac{1}{2pn+2} + \frac{1}{2pn+3} - \frac{1}{2pn+4} + \ldots + \frac{1}{2pn+2p-1} - \frac{1}{2pn+2p}$$

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$$= S_{2pn} + \frac{1}{2pn+1} - \frac{1}{2pn+2} + \frac{1}{2pn+3} - \frac{1}{2pn+4} + \ldots + \frac{1}{2pn+2p-1} - \frac{1}{2pn+2p}$$

$$= S_{2pn+2p} = S_{2p(n+1)}$$

Therefore, $f(n) = 2p(n+1)$ and $R_n = 0$.

General Case (b): $A(p,q), p > q$.

Theorem4.1.2.2. Suppose that $p > q$. (1) The last block of $C_n(p,q)$ is.

$$\left(\frac{1}{2np-2p+1} + \frac{1}{2np-2p+3} + \ldots + \frac{1}{2np-1}\right) - \left(\frac{1}{2nq-2q+2} + \frac{1}{2nq-2q+4} + \ldots + \frac{1}{2nq}\right), and$$

$$(2) C_n(p,q) = S_{2pn} + \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \ldots + \frac{1}{2np}\right).$$

*Proof*. We proceed by induction. Note first that: (Ohm, 1839)

$$C_1 = \left(1 + \frac{1}{3} + \ldots + \frac{1}{2p-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2q}\right)$$

$$C_1 = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{2p-1} - \frac{1}{2p}\right) + \left(\frac{1}{2q+2} + \frac{1}{2q+4} + \ldots + \frac{1}{2p}\right)$$

$$C_1 = S_{2p} + \left(\frac{1}{2q+2} + \frac{1}{2q+4} + \ldots + \frac{1}{2p}\right) = S_{2 \cdot 1 \cdot p} + \left(\frac{1}{2q+2} + \frac{1}{2q+4} + \ldots + \frac{1}{2p}\right)$$

$$C_2 = C_1 + \left[\left(\frac{1}{2p+1} + \frac{1}{2p+3} + \ldots + \frac{1}{4p-1}\right) - \left(\frac{1}{2q+2} + \frac{1}{2q+4} + \ldots + \frac{1}{4q}\right)\right]$$

$$C_2 = S_{4p} + \left(\frac{1}{4q+2} + \frac{1}{4q+4} + \ldots + \frac{1}{4p}\right) = S_{2 \cdot 2 \cdot p} + \left(\frac{1}{4q+2} + \frac{1}{4q+4} + \ldots + \frac{1}{4p}\right)$$

This shows that $(1)$ and $(2)$ hold when $n = 1, 2$.

Now suppose that $(1)$ and $(2)$ hold for $n$. We need to show that $(1)$ and $(2)$ hold for $n+1$.

Proof of $(1)$: By the inductive hypothesis, we know that the positive terms added in the last block of $C_n(p,q)$ are

$$\left(\frac{1}{2np-2p+1} + \frac{1}{2np-2p+3} + \ldots + \frac{1}{2np-1}\right).$$

Hence, the $p$ positive terms in the last block of $C_{n+1}(p,q)$ that are added together are

$$\frac{1}{2np+1} + \frac{1}{2np+3} + \ldots + \frac{1}{2np+2p+1}.$$

Similarly, by the inductive hypothesis, we know that the $q$ terms subtracted in the last block of $C_n(p,q)$ are

$$\left(\frac{1}{2nq-2q+2} + \frac{1}{2nq-2q+4} + \ldots + \frac{1}{2nq}\right).$$

Hence, the terms substracted in $C_{n+1}(p,q)$ must be

$$\frac{1}{2nq+2} + \frac{1}{2nq+4} + \ldots + \frac{1}{2nq+2q}.$$

This proves $(1)$ by noting that:

$2(n+1)p - 2p + 1 = 2np + 1,$

$2(n+1)p - 2p + 3 = 2np + 3, \cdots, 2(n+1)p - 1 = 2np + 2p - 1$

$2(n+1)q - 2q + 2 = 2nq + 2,$

$2(n+1)q - 2q + 4 = 2nq + 4, \cdots, 2(n+1)q = 2nq + 2q.$

Note also that:

$$C_{n+1}(p,q) = C_n(p,q) + \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \ldots + \frac{1}{2np+2p-1}\right) - \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \ldots + \frac{1}{2nq+2q}\right).$$

Proof of $(2)$: By the inductive hypothesis,

$$C_n(p,q) = S_{2np} + \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \ldots + \frac{1}{2np}\right).$$

We want to show that:

$$C_{n+1}(p,q) = S_{2(n+1)p} + \left(\frac{1}{2(n+1)q+2} + \frac{1}{2(n+1)q+4} + \ldots + \frac{1}{2(n+1)p}\right).$$

From $(1)$, we have already shown that:

$$C_{n+1}(p,q)=C_n(p,q)+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np+2p-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

Since

$$C_{n+1}(p,q)=C_n(p,q)+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np+2p-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

it is enough to show that:

$$C_{n+1}(p,q)=\left[S_{2pn}+\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}\right)\right]+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np+2p-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

By definition of $S_k$, we know that:

$$S_{2(n+1)p}-S_{2np}=\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np+2p-1}\right)-\left(\frac{1}{2np+2}+\frac{1}{2np+4}+\ldots+\frac{1}{2np+2p}\right).$$

$$=\frac{1}{2np+1}-\frac{1}{2np+2}+\frac{1}{2np+3}-\frac{1}{2np+4}+\ldots+\frac{1}{2np+2p-1}-\frac{1}{2np+2p}$$

$$=\frac{1}{2np+1}-\frac{1}{2np+2}+\frac{1}{2np+3}-\ldots-\frac{1}{2np+2p}.$$

Notice that $S_{2np}$ can be written as:

$$S_{2np}=\left[S_{2(n+1)p}-\left(\frac{1}{2np+1}-\frac{1}{2np+2}+\frac{1}{2np+3}-\ldots-\frac{1}{2np+2p}\right)\right]$$

So substracting $S_{2np}$ into our expression for $C_{n+1}(p,q)$ we get

$$C_{n+1}(p,q)=\left[S_{2(n+1)p}-\left(\frac{1}{2np+1}-\frac{1}{2np+2}+\frac{1}{2np+3}-\ldots-\frac{1}{2np+2p}\right)\right]+\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}\right)+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np+2p-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

Rewriting this so that the positive and negative terms in $\left(\frac{1}{2np+1}-\frac{1}{2np+2}+\frac{1}{2np+3}-\ldots-\frac{1}{2np+2p}\right)$ are grouped together, we have

$$C_{n+1}(p,q)=\left[S_{2(n+1)p}-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np+2p-1}\right)+\left(\frac{1}{2np+2}+\frac{1}{2np+4}+\ldots+\frac{1}{2np+2p}\right)\right]+\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}\right)+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np+2p-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

Simplifying this we have:

$$C_{n+1}(p,q)=\left[S_{2(n+1)p}+\left(\frac{1}{2np+2}+\frac{1}{2np+4}+\ldots+\frac{1}{2np+2p}\right)\right]+\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

Notice by expansion and simplification we get:

$$C_{n+1}(p,q)=S_{2(n+1)p}+\left(\frac{1}{2nq+2q+2}+\frac{1}{2nq+2q+4}+\ldots+\frac{1}{2np+2p}\right)$$

$$C_{n+1}(p,q)=S_{2(n+1)p}+\left(\frac{1}{2(n+1)q+2}+\frac{1}{2(n+1)q+4}+\ldots+\frac{1}{2(n+1)p}\right).$$

Which is what is wanted.

General Case (b): $A(p,q),p<q$.

Thus, we come up with the following theorem:

Theorem4.1.2.3. Suppose that $ p<q $. (1) The last block of $C_n(p,q)$ is

$$\left(\frac{1}{2np-2p+1}+\frac{1}{2np-2p+3}+\ldots+\frac{1}{2np-1}\right)-\left(\frac{1}{2nq-2q+2}+\frac{1}{2nq-2q+4}+\ldots+\frac{1}{2nq}\right), and$$

$$(2)C_n(p,q)=S_{2nq}\pm\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right).$$

*Proof*. We proceed by induction. Note first that: (Ohm, 1839)

$$C_1=\left(1+\frac{1}{3}+\ldots+\frac{1}{2p-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2q}\right)$$

$$=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2q-1}-\frac{1}{2q}\right)\pm\left(\frac{1}{2p+1}+\frac{1}{2p+3}+\ldots+\frac{1}{2q-1}\right)$$

$$=S_{2q}\pm\left(\frac{1}{2p+1}+\frac{1}{2p+3}+\ldots+\frac{1}{2q-1}\right)=S_{2\cdot 1\cdot q}\pm\left(\frac{1}{2p+1}+\frac{1}{2p+3}+\ldots+\frac{1}{2q-1}\right)$$

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$$C_2 = C_1 + \left[\left(\frac{1}{2p+1}+\frac{1}{2p+3}+\ldots+\frac{1}{4p-1}\right)-\left(\frac{1}{2q+2}+\frac{1}{2q+4}+\ldots+\frac{1}{4q}\right)\right]$$
$$=\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{4q-1}-\frac{1}{4q}\right)\pm\left(\frac{1}{4p+1}+\frac{1}{4p+3}+\ldots+\frac{1}{4q-1}\right)$$
$$=S_{4q}\pm\left(\frac{1}{4p+1}+\frac{1}{4p+3}+\ldots+\frac{1}{4q-1}\right)=S_{2\cdot 2\cdot q}\pm\left(\frac{1}{4p+1}+\frac{1}{4p+3}+\ldots+\frac{1}{4q-1}\right).$$

This shows that $(1)$ and $(2)$ hold when $n=1,2$.

Suppose that $(1)$ and $(2)$ hold for $n$. We need to show that $(1)$ and $(2)$ hold for $n+1$.

Proof of $(1)$: By the inductive hypothesis, we know that the positive terms added in the last block of $C_n(p,q)$ are:

$$\frac{1}{2np-2p+1}+\frac{1}{2np-2p+3}+\ldots+\frac{1}{2np-1}.$$

Since we need to add $p$ positive terms, then the $p$ positive terms in the last block of $C_{n+1}(p,q)$ that are added together are:

$$\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np+2p-1}.$$

Similarly, by inductive hypothesis, we know that the $q$ terms subtracted in the last block of $C_n(p,q)$ are:

$$\frac{1}{2nq-2q+2}+\frac{1}{2nq-2q+4}+\ldots+\frac{1}{2nq},$$

and therefore the new terms subtracted in $C_{n+1}(p,q)$ must be:

$$\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}.$$

$(1)$ follows from noting that:

$$2(n+1)p-2p+1=2np+1,$$
$$2(n+1)p-2p+3=2np+3,\cdots,2(n+1)p-1=2np+2p-1$$
$$2(n+1)q-2q+2=2nq+2,$$
$$2(n+1)q-2q+4=2nq+4,\cdots,2(n+1)q=2nq+2q.$$

Also,

$$C_{n+1}(p,q)=C_n(p,q)+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np+2p-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

Proof of $(2)$: By the inductive hypothesis,

$$C_n(p,q)=S_{2nq}\pm\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right).$$

We want to show that:

$$C_{n+1}(p,q)=S_{2(n+1)q}\pm\left(\frac{1}{2(n+1)p+1}+\frac{1}{2(n+1)p+3}+\ldots+\frac{1}{2(n+1)q-1}\right).$$

From $(1)$, we have already shown that:

$$C_{n+1}(p,q)=C_n(p,q)+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np+2p-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

Since

$$C_{n+1}(p,q)=C_n(p,q)+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np+2p-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

It suffices to show that:

$$C_{n+1}(p,q)=\left[S_{2nq}\pm\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right)\right]+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

By definition of $S_k$, we know that:

$$S_{2(n+1)q}-S_{2nq}=\left(\frac{1}{2nq+1}+\frac{1}{2nq+3}+\ldots+\frac{1}{2nq+2q-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

Notice that $S_{2nq}$ can be written as:

$$S_{2nq}=\left[S_{2(n+1)q}-\left(\frac{1}{2nq+1}-\frac{1}{2nq+2}+\frac{1}{2nq+3}-\ldots-\frac{1}{2nq+2q}\right)\right]$$

So, substracting $S_{2nq}$ into our expression for $C_{n+1}(p,q)$, we have that:

$$C_{n+1}(p,q)=\left[S_{2(n+1)q}-\left(\frac{1}{2nq+1}-\frac{1}{2nq+2}+\frac{1}{2nq+3}-\ldots-\frac{1}{2nq+2q}\right)\right]-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right)+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

Rewriting this so that the positive and negative terms in

$$\left(\frac{1}{2nq+1}-\frac{1}{2nq+2}+\frac{1}{2nq+3}-\ldots-\frac{1}{2nq+2q}\right)$$

are grouped together, we have:

$$C_{n+1}(p,q)=S_{2(n+1)q}-\left(\frac{1}{2nq+1}+\frac{1}{2nq+3}+\ldots+\frac{1}{2nq+2q-1}\right)+\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right)-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right)+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np-1}\right)-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nq+2q}\right).$$

Simplifying,

$$C_{n+1}(p,q)=S_{2(n+1)q}\pm\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq+2q-1}\right)+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2np-1}\right).$$

By expansion again and simplification,

$$C_{n+1}(p,q)=$$
$$S_{2(n+1)q}\pm\left(\frac{1}{2np+2p+1}+\frac{1}{2np+2p+3}+\ldots+\frac{1}{2nq+2q-1}\right)$$
$$C_{n+1}(p,q)=$$
$$S_{2(n+1)q}\pm\left(\frac{1}{2(n+1)p+1}+\frac{1}{2(n+1)p+3}+\ldots+\frac{1}{2(n+1)q-1}\right).$$

Recall $C_n=C_n(p,q)$ to be the partial sum of $A(p,q)$, which is the result in $S$ from precisely adding $n$ blocks, where each block consist of $p$ positive terms and $q$ negative terms, consecutively. Also, recall $S_n$ to be the partial sums of usual alternating series, where $n$ denotes the terms. Call $C_n$ as $C_n=S_{f(n)}+R_n$ for some $f(n)$ and some "remainder" $R_n$.

We aim to show that $S_n$ and $C_n=S_{f(n)}+R_n$ for some $f(n)$ converging to $ln(2)+\frac{1}{2}ln\left(\frac{p}{q}\right)$. For our statement, we begin by splitting up the proof for each of the three cases. In our statement, we label each case as 1, 2, and 3, respectively.

*Proof*. 1. Consider $C_n(p,p)=S_{2np}$, where $f(n)=2np$ and $R_n=0$.

We show that $S_{f(n)}$ converges to $ln(2)$ as $n\to\infty$, and that for each $S_{k}¡^{-}$, if $n$ is defined by

$$n(p+q)\leqslant k\leqslant(n+1)(p+q),$$

Letting $r_k=S_{k}¡^{-}-C_n$, we get $r_k\to 0$ as $k\to\infty$. Note that $n\to\infty$ if $k\to\infty$, and so $S_{k}¡^{-}\to ln(2)$.

First, $r_k$ consists of a sum of $x_1+x_2+\ldots$, which is at most $p+p-1=2p-1$ terms where each term is

$$|x_i|\leqslant\frac{1}{2np+1}.$$

So, letting $n$ be that such that $2np\leqslant k\leqslant 2(n+1)p$. Then,

$$|x_1+x_2+\ldots|\leqslant\frac{2p-1}{2np+1}.$$

By taking the limit of $\frac{2p-1}{2np+1}$, we have

$$\lim_{n\to\infty}\frac{2p-1}{2np+1}=0$$

which means that these extra terms are negligible. Now, since $S_k$ is defined as the partial sums of the usual Alternating Harmonic Series, we know that

$$\lim_{n\to\infty}S_n=ln(2).$$

By the fact that $S_{2np}$ are the partial sums of the usual alternating series up to $2np$ terms, then

$$\lim_{n\to\infty}S_{2np}=ln(2).$$

## 2. Consider

$$C_n(p,q)=S_{2np}+\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}\right),$$

where $f(n)=2np$ and

$$R_n=\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}\right).$$

We have already shown from 1. that $S_{2np}$ converges to $ln(2)$ as $n\to\infty$. Now we want to show both that

$$\lim_{n\to\infty}R_n=\lim_{n\to\infty}\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}\right)=\frac{1}{2}ln\left(\frac{p}{q}\right).$$

And that for each $S_{k}¡^{-}$, if $n$ is defined by $n(p+q)\leqslant k<(n+1)(p+q)$, then, letting $r_k=S_{k}¡^{-}-C_n$, we have that get $r_k\to 0$ as $k\to\infty$, but $n\to\infty$ if $k\to\infty$, and so $S_{k}¡^{-}\to ln(2)$.

First, notice that $r_k$ consists of a sum of $x_1+x_2+\ldots$, which is at most $p+q-1$ terms where each term is

$$|x_i|\leqslant\frac{1}{2nq+2}$$

terms so for all $k$, there exist an $n$ such that $n(p+q)\leqslant k<(n+1)(p+q)$, which means

$$|x_1+x_2+\ldots|\leqslant\frac{p+q-1}{2nq+2}.$$

By taking the limit of $\frac{p+q-1}{2nq+2}$, we have

$$\lim_{n\to\infty}\frac{p+q-1}{2nq+2}=0$$

Therefore, the size of $r_k$ is 0. In other words, $r_k$ is so small we can consider these terms to be negligible. Now,

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$$\frac{1}{2}ln\left(\frac{p}{q}\right)=\frac{1}{2}ln\left(\frac{np}{nq}\right)=\frac{1}{2}\left(ln(np)-ln(nq)\right)=\frac{1}{2}\int_{nq}^{np}\frac{1}{x}dx$$

where

$$\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}<\frac{1}{2}\int_{nq}^{np}\frac{1}{x}dx$$

Also,

$$\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nqp}>\frac{1}{2}\int_{nq+1}^{np+1}\frac{1}{x}dx=$$

$$\frac{1}{2}\left(ln(np+1)-ln(nq+1)\right)$$

$$=\frac{1}{2}ln\left(\frac{np+1}{nq+1}\right).$$

Thus,

$$\frac{1}{2}\int_{nq+1}^{np+1}\frac{1}{x}dx<\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nqp}<\frac{1}{2}\int_{nq}^{np}\frac{1}{x}dx$$

or

$$\frac{1}{2}ln\left(\frac{np+1}{nq+1}\right)<\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2nqp}<\frac{1}{2}ln\left(\frac{np}{nq}\right).$$

Now taking the limits of the left and right side, applying L'Hopitals rule, we get

$$\frac{1}{2}\lim_{n\to\infty}ln\left(\frac{np+1}{nq+1}\right)=\frac{1}{2}ln\left(\frac{p}{q}\right)$$

and

$$\frac{1}{2}\lim_{n\to\infty}ln\left(\frac{np}{nq}\right)=\frac{1}{2}ln\left(\frac{p}{q}\right).$$

Therefore,

$$\frac{1}{2}\lim_{n\to\infty}ln\left(\frac{np+1}{nq+1}\right)=\frac{1}{2}ln\left(\frac{p}{q}\right)$$

$$\leqslant\lim_{n\to\infty}\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}\right)$$

$$\leqslant\lim_{n\to\infty}\frac{1}{2}ln\left(\frac{np}{nq}\right)$$

$$=\frac{1}{2}ln\left(\frac{p}{q}\right).$$

Since both limits converge to $\frac{1}{2}ln\left(\frac{p}{q}\right)$ as $n\to\infty$, then it is true that

$$\lim_{n\to\infty}R_n=\lim_{n\to\infty}\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}\right)=\frac{1}{2}ln\left(\frac{p}{q}\right).$$

We conclude that

$$\lim_{n\to\infty}C_n(p,q)=\lim_{n\to\infty}\left[S_{2np}+\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\ldots+\frac{1}{2np}\right)\right]=$$

$$ln(2)+\frac{1}{2}ln\left(\frac{p}{q}\right)$$

by the properties of convergent series.

3. Consider

$$C_n(p,q)=S_{2nq}\pm\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right),$$

where $f(n)=2nq$ and

$$R_n=-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right).$$

We have already shown from 1. that since $S_n$ is defined as the partial sums of the usual Alternating Harmonic Series, then $S_{2nq}$ are the partial sums of the usual series up to $2nq$ terms, converging to $ln(2)$ as $n\to\infty$. Now we want to show both that

$$\lim_{n\to\infty}R_n=\lim_{n\to\infty}-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right)=\frac{1}{2}ln\left(\frac{p}{q}\right),$$

And that for each $S_k$¡¯, if $n$ is defined by $n(p+q)\leqslant k<(n+1)(p+q)$, then, letting $r_k=S_k$¡¯$-C_n$, we have that $r_k\to 0$ as $k\to\infty$. (Note that $n\to\infty$ if $k\to\infty$, and so $S_k$¡¯$\to ln(2)$ as well.) First, notice that $r_k$ consists of a sum of $x_1+x_2+\ldots$, which is at most $p+q-1$ terms where each term is

$$|x_i|\leqslant\frac{1}{2np+1}$$

terms so for all $k$, there exist an $n$ such that $n(p+q)\leqslant k<(n+1)(p+q)$, which means

$$|x_1+x_2+\ldots|\leqslant\frac{p+q-1}{2np+1}.$$

By taking the limit of $\frac{p+q-1}{2np+1}$, we have

$$\lim_{n\to\infty}\frac{p+q-1}{2np+1}=0$$

Therefore, the size of $r_k$ is 0, which means that these extra terms are negligible, and so do not converge to any value.

Note that

$$\frac{1}{2}ln\left(\frac{p}{q}\right)=\frac{1}{2}ln\left(\frac{np}{nq}\right)=\frac{1}{2}\left(ln(np)-ln(nq)\right)=\frac{1}{2}\int_{nq}^{np}\frac{1}{x}dx$$

where

$$-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right)>\frac{1}{2}\int_{nq}^{np}\frac{1}{x}dx$$

Also,

$$-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right)<\frac{1}{2}\int_{nq+1}^{np+1}\frac{1}{x}dx$$

$$=\frac{1}{2}\left(ln(np+1)-ln(nq+1)\right)$$

$$=\frac{1}{2}ln\left(\frac{np+1}{nq+1}\right).$$

So,

$$\frac{1}{2}\int_{nq+1}^{np+1}\frac{1}{x}dx>-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right)>\frac{1}{2}\int_{nq}^{np}\frac{1}{x}dx$$

or

$$\frac{1}{2}ln\left(\frac{np+1}{nq+1}\right)>-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right)>\frac{1}{2}ln\left(\frac{np}{nq}\right).$$

Now taking the limits of the left and right side, applying L'Hopitals rule, we get

$$\frac{1}{2}\lim_{n\to\infty}ln\left(\frac{np+1}{nq+1}\right)=\frac{1}{2}ln\left(\frac{p}{q}\right)$$

and

$$\frac{1}{2}\lim_{n\to\infty}ln\left(\frac{np}{nq}\right)=\frac{1}{2}ln\left(\frac{p}{q}\right).$$

Then, it is true that

$$\frac{1}{2}\lim_{n\to\infty}ln\left(\frac{np+1}{nq+1}\right)=\frac{1}{2}ln\left(\frac{p}{q}\right)$$

$$\geqslant\lim_{n\to\infty}\left[-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right)\right]$$

$$\geqslant\lim_{n\to\infty}\frac{1}{2}ln\left(\frac{np}{nq}\right)$$

$$=\frac{1}{2}ln\left(\frac{p}{q}\right).$$

Since both limits converge to $\frac{1}{2}ln\left(\frac{p}{q}\right)$ as $n\to\infty$, then it is true that

$$R_n=-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right)$$

converges to $\frac{1}{2}ln\left(\frac{p}{q}\right)$ as $n\to\infty$. So, we can conclude that

$$C_n(p,q)=S_{2nq}\pm\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\ldots+\frac{1}{2nq-1}\right)$$

converges to $ln(2)+\frac{1}{2}ln\left(\frac{p}{q}\right)$ as $n\to\infty$, by the properties of convergent series.

*Proof* 8.7. *for Theorem* 4.3.1.

*Proof*. At the start of this proof, we want to make two claims before defining the process: The series is some conditionally convergent series, which we will denote as $\sum u_i$, given that this series converges to $U$ (for $u_i\in\mathbb{R}$). Then, claim (i) states that when the positive terms are rearranged while the negative terms are left in place, the rearranged series of $\sum +u_i$ converges to a sum $V\leqslant U$, and claim (ii) likewise, when the negative terms are rearranged while the positive terms are left in place, the rearranged series of $\sum -u_i$ converges to some sum $V\geqslant U$. (Sierpiński, 1911)

It is sufficient to establish only the first part of the proposal; in fact,

$u_1+u_2+u_3+\ldots(6.1)$

is a series given which is not absolute convergent, $U$ its sum, $T>U$, applying the second part of the theorem to the series $(6.1)$ and the number $V=T$ is obviously equivalent to the application of the first part of this theorem to the series

$-u_1-u_2-u_3-\ldots(6.2)$

and the number $V=-T$.

Let

$u_1+u_2+u_3+\ldots$

Be a given non-absolutely convergent series, $U$ its sum, V a number $<U$, given in advance, and let

$a_1+a_2+a_3+\ldots(6.3)$

Be the series of positive consecutive terms of the series $(6.1)$.

Set

$U-V=l,$

$a_1+a_2+a_3+\ldots+a_n=A_n,$

$A_0=0;$

We have:

$l>0,$

$\lim_{n\to\infty}a_n=\infty,$

$\lim_{n\to\infty}A_n=0.$

Modifying the order of the terms in the series $(6.3)$, we have a different series

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$c_1 + c_2 + c_3 + \ldots (6.4)$

To determine the law of formation of an explicit expression for the terms of $(6.4)$, suppose we have already determined its $n(\geqslant 0)$ first terms and set

$c_1 + c_2 + c_3 + \ldots + c_n = C_n,$

$C_0 = 0.$

Let $n$ be a positive integer or zero: only one of three following cases may occur:

I. $A_n - C_n \leqslant l$.

$\mathrm{II}^a$. $A_n - C_n > l$ and the term $a_{n+1}$ is not in the term $C_n$.

$\mathrm{II}^b$. $A_n - C_n > l$ and the term $a_{n+1}$ is in the term $C_n$.

In case I, we choose the index $r$ to be smallest, for which $a_r$ is not part of the sum $C_n$, such that

$a_r < \frac{1}{2^n} and a_r < \frac{a_{n+1}}{2}.$

Such an index $r$ exists always since

$\lim_{n\to\infty} a_n = 0.$

We will let

$c_{n+1} = a_r.$

In case $\mathrm{II}^a$, we set

$c_{n+1} = a_{n+1}.$

Now consider case $\mathrm{II}^b$. By virtue of

$A_n > C_n,$

Which shows that the sum $C_n$ cannot contain *all* the terms of the series $A_n$. Let $a_r$ be the first term of the sum $A_n$, which does not appear in the sum $C_n$. We set:

$c_{n+1} = a_r.$

The conditions presented define perfectly the series $(6.4)$, and clearly each term in the series $(6.3)$ appears once more in the series $(6.4)$, since we picked $c_{n+1}$ to be the term of $(6.3)$ not in $C_n$. I say that in *every* term in the series $(6.3)$ is in the series $(6.4)$.

Denote by $q_m$ the number that express how many indices $n \leqslant m$ for which we have case $\mathrm{II}^b$. The sum $C_{m+1}$ will be obviously contain all the terms of the sum $A_{q_m}$. If therefore we show that

$\lim_{m\to\infty} q_m = \infty,$

It will follow that $(6.4)$ contains all the terms of the series $(6.3)$.

In this case that the equation

$\lim_{m\to\infty} q_m = \infty$

Is not satisfied, the number of indices $n$ for which we have the case $\mathrm{II}^b$ is infinite; $v$ being a fixed number, we have then for $n \geqslant v$ the case I or $\mathrm{II}^a$. Suppose there exists an index $i \geqslant v$ for which we have the case $\mathrm{II}^a$; we have $A_i > C_i$. The number of terms of the sums $A_i$ and $C_i$ being the same, the first may not contain all the terms of the second except in the case $A_i = C_i$; hence, the sum $C_i$ contains terms that do not fall within the sum $A_i$.

These terms belonging to series $(6.3)$ (as all terms of the series $(6.4)$) but not included in the sum $A_i$ have in the series $(6.3)$ an index $> i$. The sum $C_i$ contains therefore some terms $a_n$ for which $n > i$. Let $j$ be the smallest index $> i$ for which $a_j$ is in the sum $C_i$. We can easily demonstrate that for the indices $i, i+1, i+2, \cdots, j-1$ we have case $\mathrm{II}^a$, while the case $\mathrm{II}^b$ will hold for the index $j$. However, this is inconsistent with the hypothesis that for $n \geqslant v$ we always have case I or case $\mathrm{II}^a$. Now this hypothesis requires that we always have case I for $n \geqslant v$. This is being admitted, we have for $n \geqslant v$ constantly

$A_n - C_n \leqslant l and c_{n+1} < \frac{a_{n+1}}{2},$

From which for all natural $x$:

$l \geqslant A_{v+x} - C_{v+x} > A_v - C_v + \frac{a_{v+1} + a_{v+2} + \ldots + a_{v+x}}{2}.$

Or:

$a_{v+1} + a_{v+2} + \ldots + a_{v+x} < 2(l - A_v + C_v),$

Which is inconsistent with the divergence of the series $(6.3)$.

We have therefore shown that

$\lim_{n\to\infty} q_m = \infty$

Moreover, we can consider also as demonstrated that the series $(6.4)$ differs from the series $(6.3)$ only by the order of its terms.

Now denote $p_m$ to be the number expressing how many indices $n \leqslant m$ there are for which case I is realized. I says that

$\lim_{m\to\infty} p_m = \infty.$

Suppose this proposal to be inaccrutate, $v$ being a fixed number, we then have case II for all $n \geqslant v$. Donote by $\tau_n$, the number of terms of the sum $A_n$ not included in the sum $C_n$.

We obviously have

$\tau_{n+1} = \tau_n or \tau_{n+1} = \tau_n - 1$

As case $\mathrm{II}^a$ or $\mathrm{II}^b$ will take place for the index $n$. As we have shown above, case $\mathrm{II}^b$ is realized for infinitely many indices; so $\tau_n$ would be negative for sufficiently large values of $n$, which is clearly absurd. The equality

$\lim_{m\to\infty} p_m = \infty.$

Is thus established.

The sequence $a_n$ tends to 0 and the sequence $c_n$ differs from the sequence $a_n$ only by the order of its terms; we therefore also have

$\lim_{n\to\infty} c_n = 0.$

Imagine a number $h$ sufficiently large so that we have

$\frac{1}{2^{h-1}} < \epsilon (6.5)$

$\epsilon$ a positive number given in advance. As we have

$\lim_{n\to\infty} a_n = 0 and \lim_{n\to\infty} c_n = 0$

And as all terms of the series $(6.4)$ are at the same time the term of the series $(6.3)$, we set the number $\epsilon$ to match a number $v$ such that the inequality $n > v$ implies the inequalities:

$a_n < \epsilon and c_i < \epsilon (Sierpi¨½ski(2))$

And that the sum $A_v$ contains all the terms of $C_h$.

On the other hand, as we have

$\lim_{m\to\infty} p_m = \infty and, \lim_{m\to\infty} q_m = \infty$

We can match to the number $v$ a number $\mu$ such as the inequality $m > \mu$ gives:

$p_m > v and q_m > v (Sierpi¨½ski(3))$

Now let $m$ be an index $> \mu$. We examine separately the case

$A_m - C_m \leqslant l$

And the case

$A_m - C_m > l$

Suppose in the first place,

$A_m - C_m \leqslant l (6.6)$

Denote by $k$ the largest index $k < m$ for which we have

$A_k - C_k > l (6.7)$

For $n = k+1, k+2, \cdots, m$, we evidently have the inequality

$A_n - C_n \leqslant l$

Which corresponds to case I; by the meaning of the symbol $q_k$ and $q_m$, we arrive at the immediate conclusion: $q_k = q_m$. On the hand, obviously $q_k \leqslant k$. We have:

$A_{k+1} - C_{k+1} = A_k - C_k - a_{k+1} - c_{k+1} > A_k - C_k - c_{k+1} > l - c_{k+1}$

By $(6.7)$; and by $(3)$:

$k \geqslant q_k = q_m > v$

Finally, by $(2)$:

$c_{k+1} < \epsilon$

Consequently:

$A_{k+1} - C_{k+1} > l - \epsilon (6.8)$

For $n = k+1, k+2, \cdots, m$, we have case I, consequently,

$A_m - C_m - (A_{k+1} - C_{k+1}) > \frac{a_{k+2} + a_{k+3} + \cdots + a_m}{2} > 0$

Hence:

$A_m - C_m > A_{k+1} - C_{k+1} > l - \epsilon$

By $(6.8)$. Following $(6.6)$, we can write:

$-\epsilon < A_m - C_m - l \leqslant 0 (6.9)$

Now let

$A_m - C_m > l (6.10)$

Denote by $k$ the largest index $< m$ for which we have

$A_k - C_k \leqslant l$

For $n = k+1, k+2, \cdots, m$, we do not have case I; therefore, $p_k = p_m$; on the other hand, evidently $p_k \leqslant k$. We have:

$A_{k+1} - C_{k+1} = A_k - C_k + a_{k+1} - c_{k+1} < A_k - C_k + a_{k+1} \leqslant l + a_{k+1}$

But by $(2)$: $a_{k+1} < \epsilon$ since

$k \geqslant p_k = p_m > v.$

We therefore have

$A_{k+1} - C_{k+1} \leqslant l + \epsilon (6.11)$

Denote by $f_i (i = 1, 2, \cdots, s)$ the indices included in between $k$ and $m$ for which case $\mathrm{II}^b$ holds; for the other indices $n$ between $k$ and $m$, we therefore have case $\mathrm{II}^b$. Now

$a_{n+1} - c_{n+1} = 0.$

It follows that

$A_m - C_m - (A_{k+1} - C_{k+1}) = \sum_{i=1}^{s} (a_{f_i+1} - c_{f_i+1}) < \sum_{i=1}^{s} a_{f_i+1} (6.12)$

If for an index $f$ we have case $\mathrm{II}^b$, the term $a_{f+1}$ is included in the sum $C_f$, then we have

$a_{f+1} = c_{g+1} or g < f.$

I say that case I will occur for the index $g$. Indeed, if

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we had case II for the index $g$, it would set

$c_{g+1} = a_{j+1} or j \leqslant g;$

However, we have

$c_{g+1} = a_{f+1} with f > g;$

We therefore have for the index $g$ case I; or

$c_{g+1} < \frac{1}{2^g}.$

For $f = f_i(i = 1, 2, \cdots, s)$, we obviously have $f_i > k > v$ and as in the sum $A_v$ are included all the terms of the sum $C_h$, the term $a_{f_i+1}$, having an index greater than $v$, cannot be in the sum $A_v$ nor $C_h$. The index $g_i + 1$ of the term

$c_{g_i+1} = a_{f_i+1}$

Is therefore greater than $h$ and it follows that: $g_i \geqslant h$. We have case $\mathrm{II}^b$ for $f_i$; consequently, as we have demonstrated above, we have

$c_{g_i+1} < \frac{1}{2^{g_i}}$

$\sum_{i=1}^{s} a_{j_i+1} = \sum_{i=1}^{s} c_{g_i+1} < \sum_{i=1}^{s} \frac{1}{2^{g_i}} (6.13)$

$g_i(i = 1, 2, \cdots,)$ represents $s$ different numbers, all $\geqslant h$.

From that we conclude,

$\sum_{i=1}^{s} \frac{1}{2^{g_i}} < \sum_{j=h}^{\infty} \frac{1}{2^j} = \frac{1}{2^{h-1}} < \epsilon (6.14)$

By $(6.5)$. Hence, by $(6.12)$, $(6.13)$, and $(6.14)$, we get that

$A_m - C_m - (A_{k+1} - C_{k+1}) < \epsilon;$

Then by $(6.10)$ and $(6.11)$:

$0 < A_m - C_m - l < 2\epsilon (6.15)$

For all $m > \mu$, we therefore have one or the other of the inequalities $(6.9)$ or $(6.15)$; so, for all $m > \mu$, we have:

$|A_m - C_m - l| < 2\epsilon,$

From which it follows immediately that

$\lim_{m \to \infty} (A_m - C_m) = l (6.16)$

Now that

$-b_1 - b_2 - b_3 - \ldots$

By the series of consecutive negative terms of the series $(6.1)$, $-B_n$ the sum of the first $n$ terms.

We have for all natural numbers $n$:

$U_n = A_{r_n} - B_{s_n},$

$r_n$ and $s_n$ being two non-decreasing sequences and such that

$\lim_{n \to \infty} r_n = \infty and \lim_{n \to \infty} s_n = \infty (6.17)$

Now form a new series

$v_1 + v_2 + v_3 + \ldots (6.18)$

By replacing each positive term $u_n = a_{r_n}$ of the series $(6.1)$ by the term $c_n = c_{r_n}$, and maintaining without any modification the negative terms of this series.

The series $(6.18)$ differs from $(6.1)$ only by the order of its positive terms. Upon designating by $V_n$ to be the sum of the first $n$ terms of the series $(6.18)$, we evidently have:

$V_n = C_{r_n} - B_{s_n} = U_n - (A_{r_n} - C_{r_n}).$

By $(6.16)$ and $(6.17)$, we have:

$\lim_{n \to \infty} (A_{r_n} - C_{r_n}) = l$

Therefore,

$\lim_{n \to \infty} V_n = \lim_{n \to \infty} U_n - \lim_{n \to \infty} (A_{r_n} - C_{r_n}) = U - l = V.$

Proof 8.8.for Theorem 4.3.2

*Proof*. Let $\mathfrak{A} = v_1, v_2, \cdots$ be a conditionally convergent sequence of vectors of $\mathbb{R}_n$, then according to Lévy and Steinitz, the set of those vectors that can be obtained as the sum of a series through rearrangement of the terms of $\mathfrak{A}$ forms a linear manifold of $\mathbb{R}_n$. (Gödel, 1995)

The theorem provided is another version of the theorem which will make the proof easier.

Call a unit vector $e$ of $\mathbb{R}_n$ a “principle vector” if, for every positive ?, the sum of the absolute values of the vectors that form an angle $<$? with $e$ is infinite.

Lemma4.3.2.1. There are finitely many principal vectors $e_1, \cdots, e_s (s > 0)$ and. positive numbers $p_1, \cdots, p_s$( all $p_k > 0$) such that $p_1 e_1 + \cdots + p_s e_s = 0$ .

Let the set of endpoints of the principal vectors starting from the origin be called $\mathfrak{N}$, and let its convex full (the smallest convex set containing $\mathfrak{N}$) be $\mathfrak{N}^c$.

$\mathfrak{N}$, and therefore $\mathfrak{N}^c$, is compact.

The origin belongs to $\mathfrak{N}^c$, because if that were not the case, there would be a supporting hyperplane through the origin (call it $H$) such that $\mathfrak{N}^c$, and therefore all principal vectors, would lie on one side of $H$. (Call it the “$\mathfrak{N}$ side of $H$”.) The sum of the absolute values of those vectors of $\mathfrak{A}$ which do not lie on the $\mathfrak{N}$ side of $H$ would then be finite, hence so would the sum of their projections onto a line $l$ perpendicular to $H$, whereas the sum of the projections of the terms that lie on the $\mathfrak{N}$ side would be infinite,

which contradicts the convergence of $\mathfrak{A}$.
The set of endpoints of those vectors that may be represented as linear combinations, with positive coefficients, of principal vectors obviously forms a convex set $\mathfrak{M}$ that contains $\mathfrak{N}$. Thus $\mathfrak{N}^c \subset \mathfrak{M}$, and therefore $0 \in \mathfrak{M}$, so that the assertation is proved.
We say a sequence $S$ is "composed" from the sequences $S_1,\cdots,S_s$ if $S_1,\cdots,S_s$ are subsequences of $S$ such that every term of $S$ occurs in exactly one of the sequences $S_k$.
Lemma4.3.2.1. Let $\mathfrak{a}_1,\cdots,\mathfrak{a}_s$ be non - zero vectors in $\mathbb{R}_n$ that do not lie in a proper linear subspace of $\mathbb{R}_n$, and for which $\mathfrak{a}_1+\cdots+\mathfrak{a}_s=0$. Furthermore, to each $\mathfrak{a}_k$ let there be assigned a sequence $S_k$ of vectors with the same direction as there be assigned a sequence $S_k$ of vectors with the same direction as $\mathfrak{a}_k, with S_k = \mathfrak{w}_1^k,\mathfrak{w}_2^k,\cdots; \lim_{i\to\infty}\mathfrak{w}_i^k=0, \sum_{i=1}^{\infty}\left|\mathfrak{w}_i^k\right|=\infty, k=1,\cdots,s;$ and let $S_0$ be a sequence of vectors,with; and let $S_0$ be a sequence of vectors,with $S_0 = w_1, w_2,\cdots, \lim_{i\to\infty} w_i = 0.$ Then for an arbitrarily given vector w of $\mathbb{R}_n$ there is a sequence $S$ composed from $S_0, S_1,\cdots,S_s$ that has w as its sum.
It suffices to prove the theorem for $w=0$, since in the general case one can adjoin $-w$ to the sequence $S_0$ and then apply the theorem for 0.
For every $w_i$ one can specify $s$ numbers $w_i^k$, $k=1,\cdots,s$ (the "coordinates" of $w_i$) in such a way that $w_i=\sum_{k=1}^{s} w_i^k\mathfrak{a}_k$ and $\lim_{i\to\infty} w_i^k = 0$.
By the "coordinates" of $\{\mathfrak{w}_i^k\}$ is to be understood the $s$-tuple of numbers whose $k^{th}$ term is $\frac{|\mathfrak{w}_i^k|}{|\mathfrak{a}_k|}$ and whose remaining terms are 0.
By the "coordinates" $\mathfrak{S}_1^i,\mathfrak{S}_2^i,\cdots,\mathfrak{S}_s^i$ of the $i^{th}$ partial sum $\mathfrak{S}^i$ of the sequence to be constructed are to be understood the sums of the corresponding coordinates of its terms, so that one always has $\mathfrak{S}^i=\sum_{k=1}^{s}\mathfrak{S}_k^i\mathfrak{a}_k$.
Let $S^i$ denote the maximum of the numbers $\mathfrak{S}_1^i,\mathfrak{S}_2^i,\cdots,\mathfrak{S}_s^i$. The sequence to be constructed is to be formed as follows: Let its first term be $w_i$ (so that $\mathfrak{S}^1 = w_1$); then from each of the sequences $\{\mathfrak{w}_i^k\}$ take enough terms so that for the new partial sum $\mathfrak{S}_k^{n_1}$ we have $\mathfrak{S}_k^{n_1} > S^1(k=1,2,\cdots,s)$. Next follows $w_2$, and so on.
Recalling that $\mathfrak{S}_k^i - \mathfrak{S}_1^i \to 0$, $k=1,\cdots,s$ (since $\mathfrak{w}_i^k \to 0$ and $w_1 \to 0$), and considering that $\mathfrak{a}_1+\cdots+\mathfrak{a}_s=0$, one easily confirms that $\sum_S v = 0$.
Lemma 2 still holds also when the sequences $S_k$ are not parallel to $\mathfrak{a}_k$, but the sequences $\mathfrak{w}_i^k-(\mathfrak{w}_i^k)_{\text{¡}}^{-}$ converge (where $(\mathfrak{w}_i^k)_{\text{¡}}^{-}$ denotes the projection of $\mathfrak{w}_i^k$ in the direction $\mathfrak{a}_k$).
For if $\mu^k, k=1,\cdots,s$, are the sums of these sequences, then, according to Lemma 2, one may form the sequence composed from $\{(\mathfrak{w}_i^k)_{\text{¡}}^{-}\}, k=1,\cdots,s$, which has $w-(\mu^1+\cdots+\mu^s)$ as its sum. If, in this sequence, $(\mathfrak{w}_i^k)_{\text{¡}}^{-}$ is replaced by $\mathfrak{w}_i^k$, then the resulting sequence converges to $w$.
Furthermore, the following is evident (Lemma 3): For each principal vector $e$ one can form a subsequence $\{\mathfrak{w}_i\}$ of $\mathfrak{A}$ such that $\sum_i|\mathfrak{w}_i|=\infty$ and $\{\mathfrak{w}_i-(\mathfrak{w}_i)_{\text{¡}}^{-}\}$ converges absolutely (where $(\mathfrak{w}_i)_{\text{¡}}^{-}$ denotes the projection of $\mathfrak{w}_i$ onto $e$).
For the proof, one need only take a sequence of positive numbers $0 < ?_i < \frac{\pi}{2}$ with bounded sum, and, for each $?_i$

, choose from within the cone with axis $e$ and opening angle $?_i$ enough terms $v_{i_1},\cdots,v_{i_m}$ from $\mathfrak{A}$ so that their sum satisfies $1<\sum_r\left|v_{i_r}\right|<2$.
Let us now assume that Steinitz's Theorem has already been proved for space of dimension lower than $n$. Let $M$ be the linear subspace spanned by the vectors, $e_1,\cdots,e_s$, whose existence is guaranteed by Lemma 1 and let $M^\perp$ be the subspace orthogonal to it. Furthermore, let $v_M$ (respectively $v_{M^\perp}$) be the projection of $v$ onto $M$ (respectively $M^\perp$) and let $L$ be the sum domain of $\{v_{M^\perp}|v\in\mathfrak{A}\}$, which, by the inductive assumption, is a linear manifold.
We shall know that a vector $\mu$ then belongs to the sum

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domain of $\mathfrak{A}$ if and only if $\mu_{M^\perp} \in L$.

The necessity of the condition is trivial. So, for a proof of the sufficiency, let $\mu$ be a vector with $\mu_{M^\perp} \in L$; then there is an arrangement of the given sequence (we assume $\mathfrak{A}$ to be one such) such that $\sum_{\mathfrak{A}} v_{M^\perp} = \mu_{M^\perp}$. According to Lemma 3, to each $e_k$ one may, in the way indicated there, specify a subsequence $S_k$ of $\mathfrak{A}$ so that no two of the sequences $S_k$ have common terms; because $\sum_{S_k} |v_{M^\perp}| < \infty$, $\sum_{S_k} v_{M^\perp}$ is absolutely convergent.

Denote by $S_0$ the subsequence of $\mathfrak{A}$ that remains when all the sequence $S_1, \cdots, S_s$ are omitted from $\mathfrak{A}$. According to Lemma 2, a sequence $S$ can be composed from $S_0, S_1, \cdots, S_s$ so that $\sum_S v_M = \mu_M$.

Then $\mathfrak{A}$ is also composed from $S_0, S_1, \cdots, S_s$. On the other hand, from those same sequences the sequences $\sum_{\mathfrak{A}} v_{M^\perp}$, $\sum_S v_{M^\perp}$ are composed as well. The sequences $\sum_{S_k} v_{M^\perp}$, $k = 1, \cdots, s$, are convergent, hence $\sum_{S_0} v_{M^\perp}$ is also convergent; and two sequences that are composed from the same convergent sequences obviously yield the same sum. Thus $\sum_S v_{M^\perp} = \sum_{\mathfrak{A}} v_{M^\perp} = \mu_{M^\perp}$.