Solving ordinary differential equations using the Taylor Series

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ABSTRACT

This essay will shed light on some basic knowledge about Taylor Series and ordinary differential equations and then detail the principle and method of using Taylor Formula to solve ordinary differential equations. After analyzing and demonstrating examples, this essay illustrates the feasibility and advantages of the Taylor Formula in solving ordinary differential equations. It points out some problems about Taylor expansion's convergence speed and computational efficiency. Finally, this paper concludes with the applications of the Taylor Formula to solve ordinary differential equations and prospects for possible future application directions.

KEYWORDS: Ordinary differential equations, Taylor Formula, Appliances, Evaluation.

INTRODUCTION

1. Background

Taylor's formula, almost one of the most important contents of advanced math since Brook Taylor published it, was used to solve complex problems. It can also be a good way to solve ordinary differential equations. Solving an initial value problem in ordinary differential equations has been a classic numerical method for many years. Since 1966, Moore[1] has applied interval analysis to control truncation errors in solving differential equations with long Taylor series. Next, Rall [2] provides other applications of the Taylor series method.

2. The Foundation of Taylor Series.

Taylor expansion fits complex curves and turns them into polynomial equations with infinite terms. It expands the objective function at a certain point, uses the function's local properties to approximate its overall properties, and constructs a new polynomial function equal to the original function. It transforms complex functions into multiple-term equations to simplify calculations and obtain more accurate results. It is based on calculating a function's derivative and higher-order derivative at a certain point. The formula is

$$f(x)=\frac{f(x_0)}{0!}+\frac{f'(x_0)}{1!}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\ldots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x)$$

where f'(x0), f''(x0), and f ^n (x0) represent the first, second, and nth-order derivatives of f(x), respectively. The most common use of Taylor's expansion is at x=0, with the formula: f(x)=f(0)+f'(0)(x-0)+……F n(0)/n!(x-0)n. To solve differential equations, the function of Taylor Series is expanding unknown functions into series form. According to the need for accuracy, we can only take the first few terms of the expansion (the more items obtained, the more accurate the result is.) so that differential equations can become a set of linear or simple forms for easy solutions.

3. The Overview of Differential Equations.

Differential equations are mathematical equations that describe the interrelationships between variables involving unknown functions and their derivatives. Differential equations are divided into two categories: ordinary differential equations and partial differential equations. Ordinary differential equations have only one unknown function, which is my topic, while partial differential equations have multiple functions for ordinary differential equations. Its general form is F(x, y, y,…,y(n)), where F is a function about x, y, y,…,y(n), and y,…,y(n) is derivative of y concerning x. In practical applications, differential equations are important and utilized in several areas, such as chemistry, physics, and biology. The solution of differential equations is abundant, with the most commonly used approximate method being Taylor expansion. This essay aims to discuss using the Taylor Formula to solve ordinary differential equations and make an evaluation of it.

4. The Step of Using Taylor Formula Solving Differential Equations.

Taylor Formula is a method that can turn unknown functions in differential equations into their series form. It can be used to solve general ordinary differential equations and more advanced differential equations. Concerning the first-order differential equation, Using the Taylor formula to solve it is relatively simple due to the only one term of expansion, which is f(x)=f(x0)+(x-x0) f'(x0). But for the more advanced differential equations, we need to calculate the higher derivative of functions and intercept the first few terms of the series to obtain an approximate answer.

The steps of using the Taylor Formula to solve differential equations are as follows.

4.1 Expand equation using Taylor Formula

Firstly, the Taylor Formula is performed on the functions appearing in ordinary differential equations at a certain point, transforming complex functions into simple polynomial sequences. Generally speaking, the more terms in the expansion, the better the approximation effect.

4.2 Determine initial conditions

Determine the initial or boundary conditions based on the application scenarios of ordinary differential equations. These conditions typically include the initial numerical value of the variable, the derivative value of the variable, and so on.

4.3 Construct an approximate answer

Using the results of the expansion of the Taylor Formula, construct an approximate solution sequence that satisfies the Taylor expansion and initial or boundary conditions.

4.4 Solving the derivative of an approximate solution sequence

Using the results of the expansion of the Taylor Formula, solving the derivative of the approximate solution sequence usually requires using derivative formulas and recursive relationships. These derivative values will be used in the next integration process.

4.5 Performing integration operations

Using the derivative numerical value obtained in the previous step, perform an integration operation on the approximate solution sequence to obtain the numerical solution of the approximate solution sequence. (Integration operations can usually be performed using numerical integration methods such as Simpson's law, Gaussian integration, etc.)

4.6 Obtaining an approximate solution

An approximate value of the original ordinary differential equation's solution can be obtained based on the numerical solution obtained through integral operation. By continuously increasing the number of terms in the Taylor expansion mentioned above, the accuracy of the solution can be gradually improved.

5. Solving ordinary differential equations

5.1 Solving simple math questions.

As a case [3], I will select a simple but common question to solve.

Here is a simple, non-homogeneous equation.

$$y'' = x + y - y2y(0) = -1, y(0) = 1 \quad (1)$$

It is a little difficult if we use conventional solutions of differential equations. So, for people with insight, it is easy to associate with the Taylor Series, which matches the form above.

$$\begin{aligned} & y'' = \mathrm{x} + \mathrm{y} - y^2 \ \mathrm{y}(0) = -1, \mathrm{y}(0) = 1 \\ & \mathrm{y} = \mathrm{y}(0) + \mathrm{y}'(0)\mathrm{x} + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots \end{aligned} \quad (2)$$

The first and second terms meet the condition constraints of the equation and are brought into the corresponding Taylor Series in the following figure.

$$\begin{aligned} & \mathrm{y}'' = x + y - y^2 \quad \mathrm{y}(0) = -1, \mathrm{y}'(0) = 1 \\ & \mathrm{y} = -1 + 1\mathrm{x} + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots \end{aligned} \quad (3)$$

But concerning second-order derivatives in the Taylor series, flexible handling is required. In the following explanation, When the second derivative of y is equal to 0, x is equal to 0.

$$\begin{aligned} & y'' = x + y - y^2 \quad y(0) = -1, y'(0) = 1 \\ & y = -1 + 1x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots \\ & y''(0) = 0 + y(0) + [y(0)]^2 \end{aligned} \quad (4)$$

From this, we obtain the numerical value for the second derivative of y.

$$\begin{aligned} & y'' = x + y - y^2 \quad y(0) = -1, y'(0) = 1 \\ & \mathrm{y} = -1 + 1\mathrm{x} + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \cdots \\ & y''(0) = 0 + (-1) - (-1)^2 = -2 \end{aligned} \quad (5)$$

Then, recalculate the numerical value for the third derivative of y in the following figure.

$$\begin{aligned} & \frac{d}{dx}\left(y''\right) = \frac{d}{dx}\left(x + y - y^2\right) \\ & y''' = 1 + y' - 2yy' \end{aligned} \quad (6)$$

Bringing in the previous data to obtain the numerical values for the third-order derivative of y.

$$y=-1+1x-x^2+\frac{2}{3}x^3+\cdots$$

$$\frac{d}{dx}\left(y''\right)=\frac{d}{dx}\left(x+y-y^2\right) \tag{7}$$

$$y'''=1+y'-2yy'$$

$$y'''(0)=1+1-2\times(-1)\times(1)=4$$

and so on, the answer to the non-homogeneous equation is obtained in the following figure, which is in the Taylor series form of y.

$$y''=x+y-y^2 \quad y(0)=-1, y'(0)=1$$

$$y=-1+1x-x^2+\frac{2}{3}x^3-\frac{1}{3}x^4+\frac{1}{5}x^5+\cdots \tag{8}$$

$$y''=x+y-y^2 \quad y(0)=-1, y'(0)=1$$

$$y=-1+1x-x^2+\frac{2}{3}x^3-\frac{1}{3}x^4+\frac{1}{5}x^5+\cdots$$

$$y(5)\approx-1+1(5)-(5)^2+\frac{2}{3}(5)^3-\frac{1}{3}(5)^4+\frac{1}{5}(5)^5 \tag{9}$$

$$y(5)\approx-68$$

5.2 The following steps[4] are derived from zhuanlan.zhihu.com/p/433180918 by Matlab Fans, published in 14 November 2021.

For initial value problems of first-order differential equations,

$$\begin{cases}\dot{\mathbf{y}}(t)=f(t,\mathbf{y}(t))\\ \mathbf{y}(t_0)=\mathbf{y}_0\end{cases} \tag{10}$$

in the above equation, t0 is the initial time(known constant); y0 initial state(known vector); f(t, y(t)) is the function of time t and state y(t)(known function).

Expand the Taylor Series of the differential equation solution y(t) near the working point and ignore higher-order terms above the second order:

$$\mathbf{y}(t)\approx\mathbf{y}(t_k)+\dot{\mathbf{y}}(t_k)(t-t_k)+\ddot{\mathbf{y}}(t_k)\frac{(t-t_k)^2}{2} \tag{11}$$

Because of $\dot{\mathbf{y}}(t)=f(t,\mathbf{y}(t))$this, the above equation can be simple to

$$\mathbf{y}(t)\approx\mathbf{y}(t_k)+\mathbf{f}(t_k,\mathbf{y}(t_k))(t-t_k)+\left.\frac{d\mathbf{f}}{dt}\right|_{(t_k,\mathbf{y}(t_k))}(t-t_k)^2 \tag{12}$$

Define step size: $h=t_{k+1}-t_k$ Calculate the numerical value of $\mathbf{y}(t_{k+1})$when $t=t_{k+1}$:

$$\mathbf{y}(t_{k+1})=\mathbf{y}(t_k)+\mathbf{f}(t_k,\mathbf{y}(t_k))\,h+\left.\frac{d\mathbf{f}}{dt}\right|_{(t_k,\mathbf{y}(t_k))}\frac{h^2}{2} \tag{13}$$

The derivative $\frac{d\mathbf{f}}{dt}$ can be simplified by taking partial derivatives for state y(t) and t, respectively.

$$\left.\frac{d\mathbf{f}}{dt}\right|_{(t_k,\mathbf{y}(t_k))}=\left.\frac{\partial\mathbf{f}}{\partial\mathbf{y}}\right|_{(t_k,\mathbf{y}(t_k))}*\mathbf{f}(t_k,\mathbf{y}(t_k))+\left.\frac{\partial\mathbf{f}}{\partial t}\right|_{(t_k,\mathbf{y}(t_k))} \tag{14}$$

Record$t(k)=t_k,\mathbf{y}(k)=\mathbf{y}(t_k)$. So, the solution is

$$\begin{cases}\mathbf{y}(k+1)=\mathbf{y}(k)+h\left[\mathbf{f}(t(k),\mathbf{y}(k))+\left.\frac{d\mathbf{f}}{dt}\right|_{(t(k),\mathbf{y}(k))}\frac{h}{2}\right]\\ \left.\frac{d\mathbf{f}}{dt}\right|_{(t(k),\mathbf{y}(k))}=\left.\frac{\partial\mathbf{f}}{\partial\mathbf{y}}\right|_{(t(k),\mathbf{y}(k))}*\mathbf{f}(t(k),\mathbf{y}(k))+\left.\frac{\partial\mathbf{f}}{\partial t}\right|_{(t(k),\mathbf{y}(k))}\\ \mathbf{y}(0)=\mathbf{y}_0\end{cases} \tag{15}$$

The above-mentioned is about the recursive form from y(k) toy(k+1). y(1), y(2), y(3), y(4),…,y(N)… can be calculated sequentially based on the recurrence relationship of the initial conditions. This discrete sequence is the numerical solution of the differential equation.

This example demonstrates the feasibility and usefulness of using the Taylor Formula to solve ordinary differential equations from the perspective of other researchers.

5.3

Repeated differentiation is described in many textbooks. However, it is useful in the context of deriving Runge-Kutta formulas [5]

Given : y' = f(t, y); by repeated differentiation :

$$y''=f\frac{\partial f}{\partial y}+f\frac{\partial f}{\partial t},$$

$$y'''=f^2\frac{\partial^{2f}}{\partial y^2}+\left\{2\frac{\partial^2 f}{\partial t\partial y}\right\}+\left(\frac{\partial f^2}{\partial y}\right)+\frac{\partial f}{\partial t}\times\frac{\partial f}{\partial y}+\frac{\partial^2 f}{\partial t^2},$$

$$y'''=f^3\frac{\partial^3 f}{\partial y^3}+f^2\left\{3f\frac{\partial^3 f}{\partial t\partial y^2}+4\frac{\partial f}{\partial y}\times\frac{\partial^2 f}{\partial y^2}\right\}+\frac{\partial f}{\partial t}\times\left(\frac{\partial f}{\partial y}\right)^2+\frac{\partial^3 f}{\partial t^3}+3\frac{\partial f}{\partial t}\times\frac{\partial^2 f}{\partial t\partial y}+\frac{\partial^2 f}{\partial t^2}\times\frac{\partial f}{\partial y}+$$

$$f\left\{\left(\frac{\partial f}{\partial y}\right)^3+5\frac{\partial^2 f}{\partial t\partial y}\times\frac{\partial f}{\partial y}+3\frac{\partial^3 f}{\partial t^2\partial y}+3\frac{\partial f}{\partial t}\times\frac{\partial^2 f}{\partial y^2}\right\}$$

... etc. (a big mess to find many terms in this way)
Knowing y(ti), we can compute
y(ti+1) = y(ti +k) = y(ti) + k y'(ti) + y''(ti)+ y'''(ti) + ...
Then select k so that the final item used is safely below the expected local error level. Repeat ti+1, and process from the initial time.

5.4 As a comparison, here is another method to solve ordinary differential equations: difference method [6].

The following application of the difference method uses first-order ordinary differential equations as an example.
Consider the following ordinary differential equation:
y'(x) = f(x, y(x), y(x0) = y0
Where f(x, y) is the given function, y(x) is the function to be solved, and x0 and y0 are the initial conditions.
We need to discretize the continuous time axis to solve this problem using the difference method. Assuming we divide the timeline into N cells with a length of h and a time step of x=nh, then:
y (x+h)=y (x)+hy '(x)+1/2h ^ 2 * y "' (x)+.. (Taylor expansion)
Since we only consider the first derivative, we can simplify the above formula as follows:
y(x + h) = y(x) + h*y'(x)
Next, we can use the given initial conditions and the above difference formula to obtain the numerical solution of y(x) through iterative calculation. Specifically, we can make x=x0 and y(x0) = y,
Then use the difference formula to gradually calculate y(x1), y(x2), y(x3) ...y(xN).
Compared to other methods to solve ordinary differential equations, like the difference method, we can find that the Taylor Formula has advantages such as higher accuracy and clear mathematical significance. However, it is not as good as the difference method regarding computational complexity and stability. Consequently, When choosing a numerical solution, choosing based on specific problems and calculation conditions is necessary.

6 Convergence and Error Control of Taylor Series.

After the Taylor Series expansion, certain convergence conditions must be met to ensure convergence. Generally speaking, if the convergence order of the series selected for the expansion point is higher, the accuracy of expansion is higher. However, we must control the errors because the Taylor Series is the sum of a series of countless terms. The common error estimation methods include convergence order estimation, error estimation, and rapid prototyping.

7 Numerical Implementation and Computational Efficiency.

We need to compile the corresponding program code to use the Taylor Formula to solve ordinary differential equations. In the calculation process, I suggest we pay attention to the following points to improve efficiency.

7.1 Select efficient algorithms

Concerning complex problems, we need to design high-quality and efficient algorithms to improve our efficiency.

7.2 Utilize optimized technology

Take advantage of techniques such as vectorization and parallelization to optimize algorithms.

7.3 Use a suitable structure of data

Using a suitable data structure to store and access data can reduce the calculation time.

8 Error Analysis and Avoidance Methods

Some errors may occur when using the Taylor Formula to love ordinary differential equations. Common errors include rounding errors, truncation errors, initialization errors, etc. To reduce the occurrence of errors, the following measures can be taken:

8.1 Examine the logical mistakes in the process of calculating

When you are compiling the code, we should ensure the logical correctness of the calculation process.

8.2 Unit testing

For each calculation step, unit testing is required to ensure its correctness. Meanwhile, for some complex problems, numerical simulations can be used to verify.

8.3 Using fault-tolerant algorithms

For some error-prone links, fault-tolerant algorithms can be used to reduce the occurrence of errors.

9 Conclusion and Evaluation

This paper researches how to use Taylor Formula to solve ordinary differential equations. Through theoretical analysis and demonstration of examples, it illustrates the advantages and feasibility of using the Taylor Formula to solve ordinary differential equations. Using the Taylor Formula can approximate the derivative and rate of change of a function, thereby helping us better understand the local behavior of the solution.
However, the Taylor Formula also has some problems, such as slow convergence speed and high computational complexity. Therefore, further future exploration of more

efficient and accurate solution methods is needed.

Reference

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2. Rall, L.B. Automatic Differentiation: Techniques and Applications Lecture Notes in Computer Science, Vol 120, Springer-Verlag, New York, 1981.
3. Example:1-9 From baijiahao.baidu.com/s?id=1657057073012577784&wfr=spider&for=pc
4. Example:10-15 from zhuanlan.zhihu.com/p/433180918 by Matlab Fans, published on 14 November 2021.
5. Runge-Tutta Formulas: In numerical analysis, Runge-Kutta methods are an important class of implicit or explicit iterative methods for solving nonlinear ordinary differential equations. These technologies were invented by mathematicians Carl Runge and Martin Wilhelm Kutta around 1900.
6. Difference method: a numerical method for differential equations approximating derivatives through finite difference methods, thereby seeking approximate solutions to differential equations.