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Uniform Convergence of Fourier Series for Hölder Continuous Functions and Functions of Bounded Variation

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Abstract:

This article discusses two significant results about the uniform convergence of Fourier series. One is the uniform convergence of the Fourier series for Hölder-α(α>0) continuous functions, while the other is the uniform convergence of the Fourier series for functions that can be expressed as the sum of finitely many continuous monotonic functions. The two results are particularly useful. Because Hölder continuity condition ensures a specific regularity that facilitates the analysis and provides a robust framework for proving uniform convergence and many practical functions can be decomposed into monotonic components, making the theorem broadly applicable. By providing detailed proofs and applications, this study demonstrates the significant implications of uniform convergence in both theoretical and practical contexts. Moreover, this work paves the way for future research in higher-dimensional Fourier analysis, non-periodic functions, and adaptive Fourier techniques, underscoring the ongoing relevance and versatility of Fourier series in modern mathematics. Several well-known results emerge as corollaries of these findings, highlighting the robustness and applicability of these theorems in Fourier analysis.

Keywords: Fourier series, Uniform convergence, Hölder-continuous function.

1. Introduction

Fourier analysis is a cornerstone of modern mathematical analysis, serving as a fundamental tool with far-reaching applications across various scientific and engineering disciplines [1, 2]. From signal processing and heat transfer to quantum mechanics and beyond, the ability to represent functions as trigonometric series has revolutionized the way the author approach and solve complex problems. The central focus of Fourier analysis is the decomposition of functions into sums of sines and cosines, allowing for a deeper understanding of their behavior and properties. One of the most critical aspects of Fourier analysis is the convergence of Fourier series. Convergence is not merely a theoretical concern; it ensures that the Fourier series accurately approximates the target function, preserving essential properties such as continuity and integrability. This is particularly important in practical applications where precision and reliability are paramount. Uniform convergence, a stronger form of convergence, guarantees that the approximation is uniformly close to the function over its entire domain, providing a robust framework for analysis and application.

In this paper, the author explores the conditions under which Fourier series converge uniformly for specific classes of functions. It is widely known that the Fourier series of a Holder- $\alpha\left(\alpha > \frac{1}{2}\right)$ function and a bounded variation function uniformly convergent to themselves [1,2]. While in this paper, the author has improved them and present two significant results: the uniform convergence of Fourier series for Hölder- α ($\alpha > 0$) continuous functions and for functions that can be expressed as the sum of finitely many continuous monotonic functions. These results extend people's understanding of the prerequisites for uniform convergence and offer new insights into the structural properties of functions and their Fourier representations.

The proof techniques employed in this paper draw on fundamental concepts in Fourier analysis, such as the Dirichlet kernel and the Riemann-Lebesgue Lemma. These tools are essential for studying the convergence properties of Fourier series and form the backbone of the theoretical framework. By delving into these methods, this artical aim to shed light on the mechanisms that underpin uniform convergence and demonstrate its implications in both theoretical and applied contexts.

The subsequent sections of this paper will provide a detailed examination of the methods and theories that support the main results, followed by comprehensive proofs and discussions of their applications. Through this exploration, the author aims to highlight the importance of uniform convergence in Fourier analysis and its broad applicability in mathematical and physical domains. By understanding the conditions that ensure uniform convergence, the author can better harness the por of Fourier series in various fields, from theoretical research to practical problem-solving.

2. Methods and Theory

The main problem in Fourier analysis is to represent an arbitrary given function in a trigonometric series. In order to find suitable functions so that can find better approximation, it's better to concentrate the study on complex valued functions that are integrable and 2π-*periodic*. The article considers Lebesgue integrability instead of Riemann integrability which is more general and do not take account improper integral. Therefore, the standard assumption will be that $f \in \mathcal{L}[-\pi,\pi]$ and $f(-\pi)=f(\pi)$. Notice that f can be periodically extended to $\mathbb{R}$. And to be more convenient, such functions can be called to be defined on the circle [3].

2.1 Main Definition

Definition 1. *The* n^{th} *Fourier coefficient of* f *is defined by* $\hat{f}(n)=a_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,$ *and the Fourior series of* f *is* $f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}$..

Fourier series belong to a broader class of functions known as trigonometric series, which are defined as series of the form $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ If a trigonometric series has only a finite number of nonzero coefficients, it is referred to as a trigonometric polynomial, with its degree being the largest $|n|$ for $c_n \neq 0$.

Definition 2. *The* N^{th} *partial sum of Fourier series of f, for* $N \in \mathbb{N}$ *is defined by a trigonometric polynomial:* $S_N(f)(x) = \sum_{n=-N}^{N} a_n e^{inx}$

A basic question is that in what sense $S_N(f)$ converge to f as $N \to \infty$. The following so-called Parseval's identity should be familiar to us and it indicates the $\mathcal{L}^2$ convergence of $S_N(f)$.

Its proof can be seen in [4].

Theorem 1. *Let* f *be an integrable function on the circle with* $f \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$. *Then:*

1. *Mean-square convergence of the Fourier series* $\frac{1}{2\pi}\int_0^{2\pi} |f(x)-S_N(f)(x)|^2 dx \to 0$ *as* $N \to \infty$.

2. *Parseval's identity* $\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 \, dx.$

Aimed to explore deeper the properties of Fourier series, it is better to make sure that the definition is well enough. A question comes out: if f and g have the same Fourier coefficient, is f and g necessarily equal or not. See the following theorem:

Theorem 2. *if* f *and* g *are continuous and* $\hat{f}(n)=\hat{g}(n)$ *for all* $n \in \mathbb{Z}$, *then* $f=g$

Proof. By Parseval's identity $\frac{1}{2\pi}\int_0^{2\pi} |(f-g)(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} \left|\hat{f}(n) - \hat{g}(n)\right|^2 = 0.$ So $f=g$.

2.2 Convolutions

The convolution of two functions is essential in Fourier analysis and arises naturally within Fourier series.

Definition 3. *Given two* 2π*-periodic function defined on* $[-\pi,\pi]$. *The convolution of* f *and* g *is defined by:* $(f*g)(x) = \int_{-\pi}^{\pi} f(x) g(x-y) dy.$.

Loosely speaking, convolution can be treated as a "weight averages".

Proposition 3. *Suppose that* f, g *and* h *are* $2\pi-$*prriodic integrable function defined on* $[-\pi,\pi]$. *Then:*

$$f*g = g*f \quad (1)$$
$$(cf)*g = c(f*g) \quad for\ all\ c \in \mathbb{C} \quad (2)$$
$$f*(g+h) = (f*g)+(f*h) \quad (3)$$
$$(f*g)*h = f*(g*h) \quad (4)$$
$$f*g \ is\ continuous \quad (5)$$
$$\widehat{f*g}(n) = \hat{f}(n)\hat{g}(n) \quad (6)$$

Suppose $g \in C_c^k([-\pi,\pi])$ *for some* $k \geq 1$. *Then* $f*g \in C^k([-\pi,\pi])$ *and* $\frac{d^i}{dx^i}(f*g) = f*\frac{d^i g}{dx^i} \quad (\forall i = 1,\cdots,k).$ (7)

Remark 4. The first four propositions above illustrate the algebraic properties. proposition (5) shows that convolution is to some extent "more regular". proposition (6) is the key in the study of Fourier series.

The proofs are intuitional. Note that one can use continuous functions to approximate a merely integrable function [5].

2.3 Dirichlet kernel

Definition 4. *The trigonometric polynomial defined on*

$[-\pi,\pi]$ *by* $D_N(x) = \sum_{n=-N}^{N} e^{inx}$. *is called the* N^{th} *Dirichlet kernel.*

Use the summation formula for geometric progression. It can be deduced that:

$$D_N(x) = \frac{sin\left(\left(N+\frac{1}{2}\right)x\right)}{sin\left(\frac{x}{2}\right)} \quad (8)$$

The Dirichlet kernel is of great importance since the following formula for Fourier series

Proposition 5. $S_N(f)(x) = (f * D_N)(x)$

Proof. By definition and the interchangeability of a finite sum and an integral, it can be seen that:

$$S_N(f)(x) = \sum_{n=-N}^{N}\left(\int_{-\pi}^{\pi} f(y)e^{-iny}dy\right)e^{inx} =$$

$$\int_{-\pi}^{\pi} f(y)\left(\sum_{n=-N}^{N} e^{in(x-y)}\right)dy = (f * D_N)(x) \quad (9)$$

2.4 Main Proposition and Method

In demonstrating the main result in the following section, the article mainly uses two crucial propositions. For clarity, the author states them below.

Proposition 6. *Riemann-Lebesgue lemma: If* f *is an integrable function on the interval* $[0,2\pi]$ *and is* 2π*-periodic, then the Fourier sine coefficients* b_n *of* f *tend to zero as* $|n| \to \infty$.

$$\lim_{|n|\to\infty} b_n = 0 \quad (10)$$

where the Fourier sine coefficients b_n *are given by:*

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)sin(nx)dx. \quad (11)$$

It is easy to tell that this is a direct consequence of Parseval's identity (Theorem 2.1).

Proposition 7. Second Mean Value Theorem [6]

1. If the functions $f(x)$ *and* $g(x)$ *are integrable on the closed interval* $[a,b]$, *and* $f(x)$ *is a monotonic function, then there exists at least one point* ϵ *in the interval* $[a,b]$ *such that:*

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^{\varepsilon} g(x)\,dx + f(b)\int_{\varepsilon}^{b} g(x)\,dx..$$

2. If the functions $f(x)$ *and* $g(x)$ *are integrable on the closed interval* $[a,b]$, $f(x) \geqslant 0$, *and* $f(x)$ *is a monotonically decreasing function, then there exists at least one point* ϵ *in the interval* $[a,b]$ *such that:*

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^{\varepsilon} g(x)\,dx.$$

3. If the functions $f(x)$ *and* $g(x)$ *are integrable on the closed interval* $[a,b]$, $f(x) \geqslant 0$, *and* $f(x)$ *is a monotonically increasing function, then there exists at least one point* ϵ *in the interval* $[a,b]$ *such that:*

$$\int_a^b f(x)g(x)\,dx = f(b)\int_{\varepsilon}^{b} g(x)\,dx.$$

These two propositions will be used repeatedly in the next section.

3. Results and Application

It is difficult to see whether the Fourier series of a given integrable function converge to the function. However, with specific condition added, the problem allows for a more systematic study of its properties.

Definition 5. Hölder- α Continuity: *A function* f *defined on the circle is said to be Hölder* α*-continuous if there exists a constant* $C > 0$ *such that, for any two points* x, y *in* $[-\pi,\pi]$, *the following Hölder condition holds:* $|f(y) - f(x)| \le C \cdot |y - x|^{\alpha}$. *Here,* $\alpha > 0$ *is the Hölder exponent.*

Particularly, Hölder- 1 continuous functions is called to be Lipschitz continuous. Note that a C^1 function defined on the circle is automatically Hölder- 1 continuous which is Lipschitz continuous. One of the main theorem in this article is the following theorem.

Theorem 8. *Suppose a function* f *defined on the circle is Hölder* α *continuous for some* $\alpha > 0$ *then the Fourier series of* f *is converges uniformly to* f.

Proof.

Lemma 9. (pointwise convergence) *Suppose there exist* $\alpha > 0$ *and* $C > 0$ *such that for all* $y \in [-\pi,\pi]$ *then* $S_N(f)(x_0)$ *converges to* $f(x_0)$

Proof of the lemma

$$S_N(f)(x_0) - f(x_0) = (f * D_N)(x_0) - f(x_0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y)f(x_0 - y)dy - f(x_0)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(y)[f(x_0 - y) - f(x_0)]dy =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\sin\left(\left(N+\frac{1}{2}\right)y\right)\frac{f(x_0-y)-f(x_0)}{\sin\frac{y}{2}}dy \quad (12)$$

where the third equation holds by the fact that $\frac{1}{2\pi}\int_{-\pi}^{\pi}D_N(x)dx=1$. Choose $\delta>0$ Denote:

$$I_1:\frac{1}{2\pi}\int_{-\pi}^{-\delta}sin\left(\left(N+\frac{1}{2}\right)y\right)\frac{f(x_0-y)-f(x_0)}{sin\frac{y}{2}}dy \quad (13)$$

$$I_2:\frac{1}{2\pi}\int_{-\delta}^{\delta}sin\left(\left(N+\frac{1}{2}\right)y\right)\frac{f(x_0-y)-f(x_0)}{sin\frac{y}{2}}dy \quad (14)$$

$$I_3:\frac{1}{2\pi}\int_{\delta}^{\pi}sin\left(\left(N+\frac{1}{2}\right)y\right)\frac{f(x_0-y)-f(x_0)}{sin\frac{y}{2}}dy \quad (15)$$

Then, by Riemann-Lesbgue lemma I_1, I_3 $\to 0$ as $N\to\infty$. Consider

$$|I_2|=\frac{1}{2\pi}\int_{-\delta}^{\delta}\left|sin\left(\left(N+\frac{1}{2}\right)y\right)\frac{f(x_0-y)-f(x_0)}{sin\frac{y}{2}}dy\right|\leqslant$$

$$\frac{1}{2\pi}\int_{-\delta}^{\delta}\left|\frac{C|y|^\alpha}{sin\left(\frac{y}{2}\right)}\right|dy \leq \int_{-\delta}^{\delta}\frac{C|y|^\alpha}{|y|}\frac{\pi}{2}dy \leq \tilde{C}\int_0^{\delta}y^{\alpha-1}$$

$$dy=\tilde{\tilde{C}}\delta^\alpha \quad (16)$$

where $C,\tilde{C},\tilde{\tilde{C}}$ are irrelevant constants. Therefore $|I_2|\to 0$ as $\delta\to 0$. The result follows.

Back to the theorem, since f is easily seen to be continuous, $\|f\|_\infty$ exists and is finite because f is 2π-periodic. Denote $M=\|f\|_\infty$. Let $x_0\in[-\pi,\pi]$ be arbitrary and fixed. Let $\delta\in(0,1)$ be arbitrary. Recall that in the proof of the lemma, it is been denoted that

$$S_N(f)(x_0)-f(x_0)=I_1+I_2+I_3 \quad (17)$$

where $|I_2|\leqslant C\delta^\alpha$ for some constant C. irrelevant to the chosen x_0 and N and tend to zero as $\delta\to 0$. Recall that

$$I_1:\frac{1}{2\pi}\int_{-\pi}^{-\delta}sin\left(\left(N+\frac{1}{2}\right)y\right)\frac{f(x_0-y)-f(x_0)}{sin\frac{y}{2}}dy \quad (18)$$

$$I_3:\frac{1}{2\pi}\int_{\delta}^{\pi}sin\left(\left(N+\frac{1}{2}\right)y\right)\frac{f(x_0-y)-f(x_0)}{sin\frac{y}{2}}dy \quad (19)$$

By symmetry, it suffices to only focus on I_3. While

$$I_3=\frac{1}{2\pi}\int_{\delta}^{\pi}sin\left(\left(N+\frac{1}{2}\right)y\right)\frac{f(x_0-y)}{sin\frac{y}{2}}dy-\frac{f(x_0)}{2\pi} \quad (20)$$

$$\int_{\delta}^{\pi}D_N(y)dy:=U+V$$

where

$$U=\frac{1}{2\pi}\int_{\delta}^{\pi}sin\left(\left(N+\frac{1}{2}\right)y\right)\frac{f(x_0-y)}{sin\frac{y}{2}}dy \quad (21)$$

$$V=-\frac{f(x_0)}{2\pi}\int_{\delta}^{\pi}D_N(y)dy \quad (22)$$

Obviously $|V|\leqslant\frac{M}{2\pi}\left|\int_{\delta}^{\pi}D_N(y)dy\right|$, which tends to zero as $N\to\infty$ for any fixed $\delta\in(0,\pi)$ by Riemann-Lebesgue lemma and are irrelevant to chosen x_0.

To study U the author introduce function g on $[\delta,\pi]$ defined by $g(y)=\frac{f(x_0-y)}{sin\frac{y}{2}}$. Note $M_1:=\|g\|_\infty\leqslant\frac{M}{\frac{sin\delta}{2}}$ and for any $y_1,y_2\in[\delta,\pi]$ on has

$$|g(y_1)-g(y_2)|\leqslant\left|\frac{f(x_0-y_1)-f(x_0)-y_2}{sin\frac{y_1}{2}}\right|+$$

$$\left|\frac{f(x_0-y_2)}{sin\frac{y_1}{2}}-\frac{f(x_0-y_2)}{sin\frac{y_2}{2}}\right|\leqslant\frac{A|y_1-y_2|^\alpha}{sin\frac{\delta}{2}}+$$

$$M\frac{\left|\frac{y_1}{2}-\frac{y_2}{2}\right|}{\left(sin\frac{\delta}{2}\right)^2}\leqslant A|y_1-y_2| \quad (23)$$

where A and $\tilde{A}=\tilde{A}(M,\delta)$ are some positive constant depending only on M and δ which is irrelevant to chosen x_0. Choose the smallest $N_1\in\mathbb{N}$ and the largest $N_2\in\mathbb{N}$ such that

$$\bigcup_{k=N_1}^{N_2}I_K:=\bigcup_{k=N_1}^{N_2}\left[\frac{2k\pi}{N+\frac{1}{2}},\frac{2(k+1)\pi}{N+\frac{1}{2}}\right]\subset[\delta,\pi]. \quad (24)$$

Obviously, $N_2-N_1+1\leqslant\frac{\frac{\pi-\delta}{2\pi}}{N+\frac{1}{2}}\leqslant\frac{N+\frac{1}{2}}{2}\leqslant N$, and

$$\left|U-\frac{1}{2\pi}\sum_{k=N_1}^{N_2}\int_{I_k} sin\left(\left(N+\frac{1}{2}\right)y\right)g(y)dy\right|\leqslant\|g\|_\infty\cdot 2\cdot \frac{2\pi}{N+\frac{1}{2}}\leqslant\frac{4\pi}{N}M_1 \tag{25}$$

For each $k=N_1,\cdots,N_2$ the author apply mean value theorem to bound

$$\left|\int_{I_k} sin\left(\left(N+\frac{1}{2}\right)y\right)g(y)dy\right| \tag{26}$$

above by

$$\int_0^{\frac{\pi}{N+\frac{1}{2}}} sin\left(\left(N+\frac{1}{2}\right)y\right)dy\cdot C\left(\frac{2\pi}{N+\frac{1}{2}}\right)^\alpha \tag{27}$$

which is further bounded above by $\tilde{\tilde{C}}\frac{1}{N}\frac{1}{N^\alpha}$. For some constant $\tilde{C},\tilde{\tilde{C}}$ Therefore it is finally known to us that:

$$U\leqslant\frac{4\pi}{N}M_1+\frac{C}{N^\alpha} \tag{28}$$

Combined all results above,

$$S_N(f)(x_0)-f(x_0) \tag{29}$$

has an upper bound irrelevant to chosen x_0 and tends to 0 . Therefore the convergence is uniform. The theorem has now been proved.

The theorem above has many corollary and application. One is a famous theorem proved by Dirichlet which states that any C^1 function defined on the circle has uniform convergent Fourier series converged to itself.

Uniform convergence of Fourier series: main result 2

Theorem 10. *Let f be a 2π-periodic function on $\mathbb{R}$ (defined on the circle). When restricted to $[-\pi,\pi]$ (or $[0,2\pi]$,) it is a sum of finitely many continuous monotonic functions. Then $S_N(f)$ converge to f uniformly on $\mathbb{R}$ as N tends to infinity.*

Remark 11. *finitely many actually means two(2) because the sum of two continuous monotonically increasing function is still a monotonically increasing function.*

Claim: The restriction of f on $[-\pi,3\pi]$ is a sum of 4 continuous monotonic functions.

Proof of Claim: Without loss of generosity, the author may assume $f=g_1-g_2$ on $[-\pi,\pi]$ where, g_1 and g_2 are continuous monotonically increasing and $g_1(-\pi)=g_2(-\pi)=0,\ g_1(\pi)=g_2(\pi)\geqslant 0.$

Then Define

$$\widetilde{g_l}(x)=\begin{cases}\widetilde{g_l}(x)=g_l(x) & x\in[-\pi,\pi]\\ \widetilde{g_l}(x)=g_l(\pi) & x\in[\pi,3\pi]\end{cases} \tag{30}$$

for $l=1,2$. And

$$\widetilde{g_l}(x)=\begin{cases}\widetilde{g_l}(x)=0 & x\in[-\pi,\pi]\\ \widetilde{g_l}(x)=g_{l-2}(x) & x\in[\pi,3\pi]\end{cases} \tag{31}$$

for $l=3,4$. Then $f=\widetilde{g_1}-\widetilde{g_2}+\widetilde{g_3}-\widetilde{g_4}$.

Proof of the theorem: According to Claim, there exist continuous monotonic function g_1 and g_2 on $[-\pi,3\pi]$ such that $f=g_1-g_2$ on $[-\pi,3\pi]$. Denote $M=sup_{x\in[-\pi,3\pi]}(|g_1(x)|+|g_2(x)|)\leqslant\infty$. Let $x\in[0,2\pi]$ and $\delta\in(0,\pi)$ be arbitrary. Then

$$S_N(f)(x)-f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} sin\left(\left(N+\frac{1}{2}\right)y\right)\frac{f(x-y)-f(x)}{sin\frac{y}{2}}dy=\frac{1}{2\pi}\left(\int_{-\pi}^{-\delta}+\int_{-\delta}^{0}+\int_0^\delta+\int_\delta^\pi\right)\left(sin\left(N+\frac{1}{2}\right)y\right)\frac{f(x-y)-f(x)}{sin\frac{y}{2}}dy:=I_1+I_2+I_3+I_4 \tag{32}$$

By symmetry, this artical only considers I_3 and I_4. Note that

$f(x-y)-f(x)=(g_1(x-y)-g_2(x-y))-(g_1(x)-g_2(x))$

for any $x\in[0,2\pi]$ and $y\in[-\pi,\pi]$. Thus by applying the second mean-value theorem,

$$|I_4|\leqslant\frac{1}{2\pi}\cdot 4\cdot 2M\cdot sup_{[a,b]\subset[\delta,\pi]}\int_a^b D_N(y)dy \tag{33}$$

which tend to zero as N tend to zero irrelevant to x for fixed δ. Similarly, write

$f(x-y)-f(x)=(g_1(x-y)-g_1(x))-(g_2(x-y)-g_2(x))$.

And deduce from the second mean-value theorem to get

$$|I_3|\leqslant\frac{1}{2\pi}(|g_1(x-\delta)-g_1(x)|+|g_2(x_\delta)-g_2(x)|)\cdot\sup_{[a,b]\subset[0,\delta]}\left|\int_a^b D_N(y)dy\right| \tag{34}$$

which is close enough to 0 when δ is sufficiently small for all $x\in[0,2\pi]$. Combine the results proved above. The desired uniform convergence follows.

Next, the artical introduces a type of functions of vital importance. With the theorem above essentially applied in the uniform convergence of its Fourier series.

Definition 6. *A function f defined on the interval $[-\pi,\pi]$ (or equivalently on the circle) is said to be of bounded variation if the total variation* [7] *of f over $[-\pi,\pi]$ is finite.*

The total variation of f on $[-\pi,\pi]$ is defined as:

$$V_{[-\pi,\pi]}(f) = \sup\left\{\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| : -\pi \leqslant x_0 < x_1 < \cdots < x_n \leqslant \pi\right\} \quad (35)$$

are the supremum is taken over all possible partitions $\{x_0, x_1, \ldots, x_n\}$ of the interval $[-\pi,\pi]$.

In other words, f is of bounded variation if there exists a constant M such that for any partition $\{x_0, x_1, \ldots, x_n\}$ of $[-\pi,\pi]$,

$$\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})| \leqslant M \quad (36)$$

For functions defined on the circle, consider their periodic extensions and the definition of bounded variation applies similarly within one period, $[-\pi,\pi]$.

Theorem 12. *A continuous function f defined on the interval $[-\pi,\pi]$ (or equivalently on the circle) is of bounded variation if and only if it can be expressed as the sum of two continuous monotonic functions* [8].

That is, there exist two functions f_1 and f_2, both continuous and monotonic on $[-\pi,\pi]$, such that:

$$f(x) = f_1(x) + f_2(x) \text{?forall?} x \in [-\pi,\pi] \quad (37)$$

Proof. Necessity: Given f is a continuous function of bounded variation on $[-\pi,\pi]$, can construct f_1 and f_2 as follows: $f_1(x) = \frac{1}{2}\left(V_{[-\pi,x]}(f) + f(x) + f(-\pi)\right), f_2(x) = \frac{1}{2}\left(V_{[-\pi,x]}(f) - f(x) + f(-\pi)\right).$

Here, $V_{[-\pi,x]}(f)$ denotes the total variation of f from $-\pi$ to x. Both f_1 and f_2 are continuous and monotonic functions [9].

Sufficiency: If f can be written as $f = f_1 + f_2$ where f_1 and f_2 are continuous and monotonic on $[-\pi,\pi]$, then f is of bounded variation since the total variation of f can be bounded by the total variations of f_1 and f_2 [10].

Therefore, a continuous function f is of bounded variation on $[-\pi,\pi]$ if and only if it can be decomposed into the sum of two continuous monotonic functions. By the definition and the theorem above, this artical presents a strong corollary of Theorem 3.3.

Corollary 13. Let f be a continuous function of bounded variation defined on the interval $[-\pi,\pi]$ (or equivalently on the circle). Then the Fourier series of f converges uniformly to f.

4. Conclusion

In this paper, the author examines the uniform convergence of Fourier series for two types of functions: Hölder continuous functions and functions that can be represented as the sum of a finite number of continuous monotonic functions. These findings enhance people's theoretical knowledge of Fourier series and offer practical tools for their use in various scientific and engineering fields. The uniform convergence of Fourier series is a crucial aspect of Fourier analysis, ensuring that the series approximates the function ll across its entire domain. This property is particularly important in applications where preserving the continuity and integrability of the original function is essential. By establishing conditions under which uniform convergence is guaranteed, this study contributes to the robustness and reliability of Fourier analysis as a mathematical tool. For Hölder continuous functions, the uniform convergence of Fourier series highlights the importance of regularity conditions in ensuring good approximation properties. The Hölder condition provides a specific framework that facilitates analysis and allows for a more precise understanding of how Fourier series behave for functions with a certain degree of smoothness. This result has significant implications for both theoretical research and practical applications, as many real-world functions exhibit Hölder continuity.

The exploration of functions that can be decomposed into finitely many continuous monotonic components extends the applicability of Fourier series to a broader class of functions. This result is particularly useful because many practical functions can be approximated or represented in this form. By demonstrating that the Fourier series of such functions converge uniformly, it has been provided a powerful tool for analyzing and approximating complex functions in a variety of contexts. In conclusion, the results presented in this paper reinforce the significance of uniform convergence in Fourier analysis and underscore the versatility and robustness of this mathematical tool. By establishing clear conditions for uniform convergence, the author provides a foundation for further research and application in various fields. The study of Fourier series remains a vibrant area of mathematical research, with ongoing efforts to uncover new results and applications.

Through this work, the author contributes to the rich tapestry of knowledge that makes Fourier analysis an indispensable tool in both theoretical and applied mathematics.

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